ON MODEL CATEGORIES

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Contents

1. Introduction

Topological spaces have the notion of homotopy equivalence between them, maps $f : X \to Y$ and $g : Y \to X$ which are homotopic to respective identities. We want to invert these homotopies to talk about "spaces up to homotopy" and this gives the notion of homotopy types. But there is another notion of a weak homotopy equivalence between topological spaces, $f : X \to Y$, which induces isomorphisms $\pi_n(X) \cong \pi_n(Y)$ for all n. One needs to invert these weak homotopy equivalences to talk about topological spaces up to some weak equivalence.

In the 1960s, Quillen observed one can do homotopy theory in other settings. He introduced model categories in his seminal work [Qui67]"Homotopical Algebra" to use homotopic methods in homological algebra. Model categories encapsulates and abstractifies this idea of weak equivalences to any arbitrary category and enables one to do a version of homotopy theory by categorical methods. For example in category of chain complexes, or simplicial sets, etc. Model category is a category with three specified class of morphisms; Weak equivalences $\mathcal{W}$, fibrations $\mathcal{F}ib$ and cofibrations $\mathcal{C}of$. Conceptually one should think of model categories as categories which allows one to invert the weak equivalences, where fibrations and cofibrations are technical tools which allows one to do so.

We will define model categories and see some examples. As per categorical thinking, the next natural step is to define appropriate morphisms of model categories and their equivalences. We do that in section 2. In the final section, we will see how model categories can be used to present $(\infty, 1)$ categories.

Tl;DR: Model categories model homotopy theory in an arbitrary category. They help us to have a hold on the localizations and to compute derived functors. They also present $(\infty, 1)$ categories.

Disclaimer: Parts of these lecture notes were prepared with the assistance of generative AI, which was used to convert handwritten notes into LaTeX typesetting and to help locate references and theorem statements. All content has been reviewed for accuracy; however, readers are encouraged to verify any cited results independently.

2. Model categories: Definition and Examples

Definition 2.1. Given morphisms $i : A \to B$ and $p : X \to Y$ in a category, we say i has the left lifting property (**LLP**) with respect to p and p has right lifting property (**RLP**) with respect to i if the following commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

admits a diagonal filler $d : B \to X$. That is $d \circ i = p \circ d$. We denote this by saying $i \perp p$.

For a class of maps $\mathcal{S}$, we define

$$^{\perp}S = \{i \mid i \perp f \text{ for every } f \in \mathcal{S}\}, \qquad S^{\perp} = \{p \mid f \perp p \text{ for every } f \in \mathcal{S}\}$$

A concise definition of model category can be given using weak factorization systems which was introduced by Joyal [Joy08]

Definition 2.2 (Weak Factorization System)**.** A **weak factorization system**(WFS) on $\mathcal{C}$, is a pair $(\mathcal{L}, \mathcal{R})$ of classes of map such that

(1) $\mathcal{L} =^{\perp} \mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\perp}$.
(2) any map $f \in \mathcal{C}$ factors as $f = pi$ with $i \in \mathcal{L}, p \in \mathcal{R}$.

We don't ask the factorization of the morphism to be unique.

Example 2.3. Surjections and Injections (Sur, Inj) forms a WFS on **Set**.

Proposition 2.4. *[AB25, Prop 3.4] There are precisely six weak factorization systems on the category of sets, as shown in the Hasse diagram below.*

$$\begin{tikzcd} & (\mathsf{any}, \mathsf{bij}) & \\ (\mathsf{any}_{\neq\emptyset} \cup \{\mathrm{id}_\emptyset\}, \mathsf{bij} \cup \mathsf{inj}_\emptyset) \arrow[ur] & & (\mathsf{inj}, \mathsf{surj}) \arrow[ul] \\ (\mathsf{surj}, \mathsf{inj}) \arrow[u] & & (\mathsf{inj}_{\neq\emptyset} \cup \{\mathrm{id}_\emptyset\}, \mathsf{surj} \cup \mathsf{inj}_\emptyset) \arrow[u] \arrow[ull] \\ & (\mathsf{bij}, \mathsf{any}). \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{1}$$

Definition 2.5. A **model category** is a bicomplete category with three classes of morphisms: weak equivalences $\mathcal{W}$, fibrations $\mathcal{F}ib$ and cofibrations $\mathcal{C}of$ satisfying the following axioms

(1) Weak equivalences $\mathcal{W}$ satisfy 2-out-of-3 property.
(2) The pairs $(\mathcal{C}of, \mathcal{W} \bigcap \mathcal{F}ib)$ and $(\mathcal{C}of \bigcap \mathcal{W}, \mathcal{F}ib)$ are weak factorization systems.

The class of morphisms $\mathcal{W} \bigcap \mathcal{F}ib$ will be called **acyclic fibrations** and the class $\mathcal{C}of \bigcap \mathcal{W}$ will be called **acyclic cofibration**.

Unwrapping the definition 2.5, we see define a model category more explicitly as below

Definition 2.6. A model $\mathcal{C}$ is a category with three classes weak equivalences $\mathcal{W}$, fibrations $\mathcal{F}ib$ and cofibrations $\mathcal{C}of$ which satisfies the following axioms

(1) $\mathcal{C}$ admits all small limits and colimits.
(2) Weak equivalences $\mathcal{W}$ satisfy 2-out-of-3 property.
(3) The classes $\mathcal{W}, \mathcal{F}ib, \mathcal{C}of$ are all closed under retracts.
(4) Cofibrations have LLP with respect to acyclic fibrations and fibrations have RLP with respect to acyclic cofibrations.
(5) Any map $f \in \mathcal{C}$ can be factorized in two ways
   - a cofibration followed by an acyclic fibration
   - an acyclic cofibration followed by a fibration

Definition 2.7. Given a morphism $f : A \to B$, we say f is a **retract** if there exists a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "id"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "id"'] & D \arrow[r] & B \end{tikzcd}$$

Notation 2.8. We will use the following arrow notations to denote each classes

- $\xrightarrow{\sim}$ to denote weak equivalences
- $\twoheadrightarrow$ to denote fibrations
- $\hookrightarrow$ to denote cofibrations

Cofibrations can be thought of as "nice" injections, and fibrations as "nice" surjections.

Definition 2.9. An object $X \in \mathcal{C}$ is defined to be

- **cofibrant** if the natural map $\varnothing \to X$ is a cofibration;
- **fibration** if the natural map $X \to *$ is a fibration

Even if the objects we care about are not fibrant or cofibrant, we obtain weak equivalent fibrant and cofibrant objects using the factorizations.

$$\varnothing \hookrightarrow QX \overset{\sim}{\twoheadrightarrow} X \qquad ; \qquad X \overset{\sim}{\hookrightarrow} RX \twoheadrightarrow *$$

where QX is the cofibrant object weak equivalent to X and RX is the fibrant object weak equivalent to X. These are called the cofibrant and fibrant replacement respectively.

Proposition 2.10. *Given a model category $\mathcal{C}$, we have*

- *Isomorphisms are acyclic fibrations and acyclic cofibrations.*
- *A map $i \in \mathcal{C}of$ if and only if it is a retract of another morphism in $\mathcal{C}of$.*
- *(Co)fibrations are stable under (co)base change and composition*

3. Examples of Model categories

3.1. Model structures on Cat.

Let $\mathcal{C}$ be complete and cocomplete.

Example 3.1 (Discrete model structure)**.** Take

$$\mathcal{W} = \{\text{isomorphisms}\}, \qquad \mathcal{C}of = \{\text{all maps}\}, \qquad \mathcal{F}ib = \{\text{all maps}\}.$$

Then $\mathcal{C}$ is a model category and $\mathrm{Ho}(\mathcal{C}) = \mathcal{C}$.

Example 3.2 (A coarse model structure)**.** Take

$$\mathcal{W} = \{\text{all maps}\}, \qquad \mathcal{C}of = \{\text{all maps}\}, \qquad \mathcal{F}ib = \{\text{isomorphisms}\}.$$

The homotopy category is equivalent to the terminal category when $\mathcal{C}$ is nonempty. The dual choice $\mathcal{C}of = \{\text{isomorphisms}\}$ and $\mathcal{F}ib = \{\text{all maps}\}$ also works.

Theorem 3.3 (Folk model structure on **Cat**)**.** *The category of small categories admits a model structure in which [Hov07]:*

- *weak equivalences are equivalences of categories;*
- *cofibrations are functors injective on objects;*
- *fibrations are isofibrations.*

Here $F: \mathcal{C} \to \mathcal{D}$ is an isofibration if, for every $c \in \mathcal{C}$ and every isomorphism $u: F(c) \to d$ in $\mathcal{D}$, there is an isomorphism $v: c \to c'$ in $\mathcal{C}$ with $F(v) = u$. Every category is both cofibrant and fibrant.

For the category of sets, we have the following theorem from [AB25], which they attribute to a mathoverflow exchange by Goodwillie here

Theorem 3.4 (Goodwillie)**.** *There are precisely nine model structures on the category of sets, as collected in the following table along with their homotopy categories:*

Cofibrations	*Fibrations*	*Weak equivalences*	*Homotopy category*
bij	any	any	(-2)-types
surj	inj	any	(-2)-types
$\mathsf{inj}_{\neq\emptyset} \cup \{\mathrm{id}_\emptyset\}$	$\mathsf{surj} \cup \mathsf{inj}_\emptyset$	any	(-2)-types
$\mathsf{any}_{\neq\emptyset} \cup \{\mathrm{id}_\emptyset\}$	$\mathsf{bij} \cup \mathsf{inj}_\emptyset$	any	(-2)-types
inj	surj	any	(-2)-types
any	bij	any	(-2)-types
inj	$\mathsf{surj} \cup \mathsf{inj}_\emptyset$	$\mathsf{any}_{\neq\emptyset} \cup \{\mathrm{id}_\emptyset\}$	(-1)-types
any	$\mathsf{bij} \cup \mathsf{inj}_\emptyset$	$\mathsf{any}_{\neq\emptyset} \cup \{\mathrm{id}_\emptyset\}$	(-1)-types
any	any	bij	0-types

3.2. Model structure on Topological Spaces.

Model category	**Cofibrations**	**Fibrations**	**Weak equivalences**
$\mathbf{Top}_{Quillen}$	retracts of CW complex inclusions	Serre fibrations	weak homotopy equivalences
$\mathbf{Top}_{Hurewicz}$	Hurewicz cofibration	Hurewicz fibration	homotopy equivalences

3.3. Model structure on Simplicial sets.

Model category	**Cofibrations**	**Fibrant objects**	**Weak equivalences / presented theory**
Kan–Quillen **sSet**	monomorphisms	Kan complexes	weak homotopy equivalences; presents the ∞-category of spaces
Joyal **sSet**	monomorphisms	quasi-categories	categorical equivalences; presents the ∞-category of $(\infty, 1)$-categories
Bergner **sCat**	determined by lifting / generators	simplicial categories with Kan mapping spaces	Dwyer–Kan equivalences; another presentation of $(\infty, 1)$-categories

4. Homotopy Category of a Model Category

Given a model category$\mathcal{C}$, its homotopy category is defined as the localization at the weak equivalences $\mathcal{C}[\mathcal{W}]$.

The localization of an arbitrary category $\mathcal{C}$ along a class of morphism S can be defined as the category $C[S]$, where the objects are objects of $\mathcal{C}$ and the morphisms are given by zigzags

$$\bullet \xleftarrow{\in S} \bullet \longrightarrow \bullet \xleftarrow{\in S} \bullet \longrightarrow \cdots \xleftarrow{\in S} \bullet \longrightarrow \bullet$$

These localizations created, need not even be a small category anymore and the class of morphisms can be arbitrarily large. However having a model structure on the category helps one to have a control on the localization.

4.0.1. *Cylinder and path objects.* Let X be an object of a model category. Factor the fold map as

$$X \amalg X \xrightarrow{(i_0,i_1)} \mathrm{Cyl}(X) \xrightarrow{\sim} X,$$

where (i_0, i_1) is a cofibration. Such a factorization is a *cylinder object.* If X is cofibrant, then i_0 and i_1 are trivial cofibrations: since the composite of either i_k with $\mathrm{Cyl}(X) \to X$ is id_X, two-out-of-three applies.

Dually, factor the diagonal as

$$Y \xrightarrow{\sim} \mathrm{Path}(Y) \xrightarrow{(p_0,p_1)} Y \times Y,$$

where (p_0, p_1) is a fibration. This is a *path object.* If Y is fibrant, then each p_k is a trivial fibration.

Definition 4.1. Maps $f, g\colon X \to Y$ are *left homotopic*, written $f \simeq_L g$, if there is a map $H\colon \mathrm{Cyl}(X) \to Y$ with $Hi_0 = f$ and $Hi_1 = g$:

$$\begin{tikzcd} X \arrow[r, bend left, "i_0"] \arrow[r, bend right, "i_1"'] & \mathrm{Cyl}(X) \arrow[r, "H"] & Y. \end{tikzcd}$$

They are *right homotopic*, written $f \simeq_R g$, if there is $K\colon X \to \mathrm{Path}(Y)$ with $p_0K = f$ and $p_1K = g$.

Theorem 4.2 (Quillen homotopy theorem)**.** *If X is cofibrant and Y is fibrant, left and right homotopy coincide, are independent of the chosen cylinder/path objects, and define an equivalence relation $\simeq$ on $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. This relation is compatible with composition.*

Definition 4.3. The **homotopy category** of $\mathcal{C}$, is a category $\mathrm{Ho}(\mathcal{C})$, which has the same objects as $\mathcal{C}$ and whose morphisms are given by

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(RQX, RQY)/\simeq$$

Thus a general zigzag in the localization can be represented by a single map from a cofibrant replacement to a fibrant replacement

$$X \overset{\sim}{\twoheadleftarrow} QX \xrightarrow{\tilde{f}} RY \overset{\sim}{\hookleftarrow} Y$$

Theorem 4.4 (Model-categorical Whitehead theorem)**.** *A map $f\colon X \to Y$ between cofibrant–fibrant objects is a weak equivalence if and only if it is a homotopy equivalence: there exists $g\colon Y \to X$ with*

$$gf \simeq \mathrm{id}_X, \qquad fg \simeq \mathrm{id}_Y.$$

Example 4.5 (Spaces)**.** For CW complexes, the preceding theorem recovers the classical Whitehead theorem: a weak homotopy equivalence between CW complexes is a homotopy equivalence.

The following proposition shows that the homotopy category $\mathrm{Ho}(\mathcal{C})$ that we constructed is indeed a localization at weak equivalences. Let $H : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ be the natural functor into the homotopy category, then

Proposition 4.6. *[DS95, Prop 5.8] A morphism $f \in \mathcal{C}$ is a weak equivalence if and only if $H(f)$ is an isomorphism in $\mathrm{Ho}(C)$.*

5. Quillen adjunctions and Equivalences

Definition 5.1. [Rie14, Lemma 11.3.11] Let $\mathcal{C}, \mathcal{D}$ be model categories, an adjunction between the categories

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

is called a Quillen adjunction, if any of the following equivalent conditions hold.

- F preserves cofibrations and trivial cofibrations.
- G preserves fibrations and trivial fibrations.
- F preserves cofibrations and G preserves fibrations.
- F preserves acyclic cofibrations and G preserves acyclic fibrations.

Consequently, a Quillen adjunction induces derived functors

$$\mathbb{L}F\colon \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}) : \mathbb{R}G,$$

computed by

$$\mathbb{L}F(X) = F(QX), \qquad \mathbb{R}G(Y) = G(RY).$$

These are again adjoint.

Definition 5.2. A Quillen adjunction $F \dashv G$ is a *Quillen equivalence* if, for every cofibrant $X \in \mathcal{C}$ and fibrant $Y \in \mathcal{D}$, a map

$$FX \longrightarrow Y$$

is a weak equivalence in $\mathcal{D}$ if and only if its adjoint

$$X \longrightarrow GY$$

is a weak equivalence in $\mathcal{C}$.

Theorem 5.3. *A Quillen adjunction is a Quillen equivalence if and only if the derived adjunction*

$$\mathbb{L}F\colon \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}) : \mathbb{R}G$$

is an equivalence of categories.

More strongly, a Quillen equivalence induces an equivalence of the associated $(\infty, 1)$-categories.

5.1. Examples of Quillen equivalences.

Example 5.4 (Spaces and simplicial sets)**.** The geometric realization–singular complex adjunction

$$|-|\colon \mathbf{sSet}_{\mathrm{KQ}} \rightleftarrows \mathbf{Top}_{\mathrm{Serre}} : \mathrm{Sing}$$

is a Quillen equivalence. It expresses the equivalence between combinatorial and topological models for spaces.

Example 5.5 (Dold–Kan)**.** For a ring R, the normalization functor and its inverse give an equivalence

$$N\colon \mathbf{sMod}_R \rightleftarrows \mathbf{Ch}_{\geq 0}(R) : \Gamma.$$

With the standard model structures, this is a Quillen equivalence (in fact, an equivalence of underlying categories). Homotopy groups of simplicial R-modules correspond to homology groups of normalized chain complexes.

Example 5.6 (Two models of $(\infty, 1)$-categories)**.** The coherent realization–homotopy coherent nerve adjunction

$$\mathfrak{C}[-]\colon \mathbf{sSet}_{\mathrm{Joyal}} \rightleftarrows \mathbf{sCat}_{\mathrm{Bergner}} : N_{\mathrm{hc}}$$

is a Quillen equivalence. This is one precise sense in which quasi-categories and simplicial categories describe the same $(\infty, 1)$-categorical objects.

6. From Model Category to $(\infty, 1)$-categories

Let $(\mathcal{M}, \mathcal{W})$ be a model category, regarded as a relative category by retaining its weak equivalences. The ordinary localization $\mathcal{M}[\mathcal{W}^{-1}]$ records homotopy classes of maps, but it forgets the higher homotopies between them. Dwyer–Kan simplicial localization replaces it by a simplicial category whose mapping objects model the full derived mapping spaces [DK80b; DK80a].

6.1. **Hammock localization.** The hammock localization $L^H(\mathcal{M}, \mathcal{W})$ is a simplicial category, which has the same objects as $\mathcal{M}$. For $X, Y \in \mathcal{M}$, a 0-simplex of $L^H(\mathcal{M}, \mathcal{W})(X, Y)$ is represented by a finite zigzag

$$X \xleftarrow{\sim} C_1 \longrightarrow C_2 \xleftarrow{\sim} \cdots \longrightarrow C_r \longrightarrow Y,$$

where every arrow pointing to the left lies in $\mathcal{W}$. More generally, an k-simplex is a *reduced hammock of width k*: a commutative diagram as below

$$\begin{tikzcd}
 & C_{0,1} \arrow[r] \arrow[d] \arrow[ddl, "\sim"'] & C_{0,2} \arrow[d] & \cdots \arrow[l, "\sim"'] \arrow[r] \arrow[d, dashed] & C_{0,n} \arrow[d] \arrow[ddr] & \\
 & C_{1,1} \arrow[r] \arrow[d] \arrow[dl, "\sim"'] & C_{1,2} \arrow[d] & \cdots \arrow[l, "\sim"'] \arrow[r] \arrow[d] & C_{1,n} \arrow[d] \arrow[dr] & \\
X & C_{2,1} \arrow[r] \arrow[d] \arrow[l, "\sim"'] & C_{2,2} \arrow[d] & \cdots \arrow[l, "\sim"'] \arrow[r] \arrow[d, dashed] & C_{2,n} \arrow[d] \arrow[r] & Y \\
 & \vdots \arrow[r, dashed] \arrow[d] \arrow[ul, "\sim"] & \vdots \arrow[d] & \vdots \arrow[l, dashed, "\sim"'] \arrow[r, dashed] \arrow[d, dashed] & \vdots \arrow[d] \arrow[ur] & \\
 & C_{k,1} \arrow[r] \arrow[uul, "\sim"] & C_{k,2} & \cdots \arrow[l, "\sim"'] \arrow[r] & C_{k,n} \arrow[uur] &
\end{tikzcd}$$

with $k + 1$ such rows, fixed endpoints X and Y, and vertical arrows in $\mathcal{W}$.We need the reduced hammock to satisfy

- the length of the hammock $n \geq 0$
- all vertical maps are weak equivalences
- all maps go in the same direction in a column, in particular if they go to left, they are all in $\mathcal{W}$
- maps in adjacent column go in opposite direction
- no column contains only identity maps.

The basic comparison theorem states that

$$\pi_0 L^H(\mathcal{M}, \mathcal{W})(X, Y) \cong \mathrm{Hom}_{\mathcal{M}[\mathcal{W}^{-1}]}(X, Y),$$

Thus the hammock localization recovers the ordinary homotopy category after applying π_0, while retaining all higher homotopy information.

6.2. **The homotopy coherent nerve.** For $n \geq 0$, let $\mathfrak{C}[\Delta^n]$ be the simplicial category with objects $0, \ldots, n$. For $i \leq j$, let

$$P_{i,j} = \{S \subseteq \{i, i+1, \ldots, j\} \mid i, j \in S\},$$

ordered by inclusion, and set

$$\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i, j) = N(P_{i,j}), \qquad \mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i, j) = \varnothing \quad (i > j).$$

Composition is induced by union of subsets. Since $P_{i,j}$ is the Boolean lattice on the interior vertices $i+1,\ldots,j-1$, one has

$$N(P_{i,j}) \cong (\Delta^1)^{j-i-1} \qquad (i<j).$$

The homotopy coherent nerve is a functor

$$N_{hc} : \mathbf{sCat} \to \mathbf{sSet}$$
$$\mathcal{C} \mapsto \mathrm{Hom}_{\mathbf{sCat}}(\mathfrak{C}[\cdot], \mathcal{C})$$

Thus the homotopy coherent nerve of a simplicial category $\mathcal{C}$ is [Lur09, §1.1.5]

$$N_{\mathrm{hc}}(\mathcal{C})_n := \mathrm{Hom}_{\mathbf{sCat}}\big(\mathfrak{C}[\Delta^n], \mathcal{C}\big).$$

By definition and Kan extension, this lifts to an adjunction

$$\mathrm{Hom}_{\mathbf{sSet}}(S, N(\mathcal{C})) = \mathrm{Hom}_{\mathbf{sCat}}(\mathfrak{C}[S], \mathcal{C})$$

A 1-simplex is an enriched morphism. A 2-simplex consists of morphisms f_{01}, f_{12}, f_{02} together with a specified 1-simplex in $\mathrm{Map}_{\mathcal{C}}(x_0, x_2)$ from f_{02} to $f_{12}f_{01}$. A 3-simplex supplies a homotopy between the two resulting homotopies, and higher simplices encode the corresponding higher coherence. If the mapping simplicial sets of $\mathcal{C}$ are Kan complexes, then $N_{\mathrm{hc}}(\mathcal{C})$ is a quasicategory.

The underlying ∞-category. For a general model category one may define

$$\mathcal{M}_\infty := N_{\mathrm{hc}}\big(L^H(\mathcal{M}, \mathcal{W})^{\mathrm{fib}}\big),$$

where $(-)^{\mathrm{fib}}$ denotes a fibrant replacement in the Bergner model structure on simplicial categories. If $\mathcal{M}$ is a simplicial model category and $\mathcal{M}^{cf}$ is its full simplicial subcategory of cofibrant–fibrant objects, then its mapping spaces are Kan complexes and the canonical comparison is a Dwyer–Kan equivalence. Consequently,

$$\boxed{\mathcal{M}_\infty \simeq N_{\mathrm{hc}}(\mathcal{M}^{cf})}, \qquad \mathrm{Map}_{\mathcal{M}_\infty}(X,Y) \simeq \mathrm{Map}_{\mathcal{M}}(X,Y) \quad (X, Y \in \mathcal{M}^{cf}).$$

Thus simplicial localization performs the homotopical inversion of weak equivalences, while the homotopy coherent nerve repackages the resulting simplicial category as an ∞-category.

References

[AB25] Omar Antolín-Camarena and Tobias Barthel. *The Nine Model Category Structures on the Category of Sets*. Aug. 21, 2025. DOI: 10.48550/arXiv.2508.15731. arXiv: 2508.15731 [math.CT]. Pre-published (cit. on pp. 2, 4).

[DK80a] W.G. Dwyer and D.M. Kan. "Calculating Simplicial Localizations". In: *Journal of Pure and Applied Algebra* 18.1 (July 1980), pp. 17–35. ISSN: 00224049. DOI: 10.1016/0022-4049(80)90113-9 (cit. on p. 7).

[DK80b] W.G. Dwyer and D.M. Kan. "Simplicial Localizations of Categories". In: *Journal of Pure and Applied Algebra* 17.3 (June 1980), pp. 267–284. ISSN: 00224049. DOI: 10.1016/0022-4049(80)90049-3 (cit. on p. 7).

[DS95] W. G. Dwyer and J. Spalinski. "CHAPTER 2 - Homotopy Theories and Model Categories". In: *Handbook of Algebraic Topology*. Ed. by I. M. James. Amsterdam: North-Holland, Jan. 1, 1995, pp. 73–126. ISBN: 978-0-444-81779-2. DOI: 10.1016/B978-044481779-2/50003-1 (cit. on p. 6).

[Hov07] Mark Hovey. *Model Categories*. Vol. 63. Mathematical Surveys and Monographs. Providence, Rhode Island: American Mathematical Society, Oct. 17, 2007. ISBN: 978-0-8218-4361-1 978-1-4704-1290-6. DOI: 10.1090/surv/063 (cit. on p. 4).

[Joy08] André Joyal. *The theory of quasi-categories and its applications*. Lecture notes, Centre de Recerca Matemàtica, Barcelona. 2008. URL: https://ncatlab.org/nlab/files/JoyalTheoryOfQuasiCategories.pdf (cit. on p. 2).

[Lur09] Jacob Lurie. *Higher Topos Theory (AM-170):* Princeton University Press, Dec. 31, 2009. ISBN: 978-1-4008-3055-8. DOI: 10.1515/9781400830558 (cit. on p. 8).

[Qui67] Daniel G. Quillen. *Homotopical Algebra.* Red. by A. Dold and B. Eckmann. Vol. 43. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer, 1967. ISBN: 978-3-540-03914-3 978-3-540-35523-6. DOI: 10.1007/BFb0097438 (cit. on p. 1).

[Rie14] Emily Riehl. *Categorical Homotopy Theory.* New Mathematical Monographs 24. New York: Cambridge university press, 2014. ISBN: 978-1-107-04845-4 (cit. on p. 6).