Kan we fill the outer horns?

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Introduction

Overview

- Joyal lifting theorem states that given an inner Kan fibration, one can fill *special* outer horns.
- One main corollary is that a Right Kan simplicial set is Kan simplicial.
- Our objective is to generalize this result to an arbitrary category with a Grothendieck pretopology.

Upshot

Assume Right Kan conditions imply Kan conditions.

Let $M \xrightarrow{\chi} \Delta^1$ be a right Kan fibration, between any two simplicial object in the category $(\mathcal{C}, \mathcal{T})$, then both the end points $\chi^{-1}\{0\}$ and $\chi^{-1}\{1\}$ are right Kan fibrant and hence by the assumption ∞-groupoids.

Joyal Lifting theorem

Joyal Lifting theorem

Theorem

Let $p : X \to Y$ be an inner fibration between ∞-categories and $f \in X$ such that $p(f)$ is an isomorphism in Y. Then TFAE:

- *f is an isomorphism in X.*
- *For all $n \geq 2$, there is a lift for the diagram*

$$\begin{array}{ccc} \Lambda^n_0 & \xrightarrow{h} & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

where $h|_{\Delta^1\{0,1\}}$ is f.

Joyal Lifting theorem

A left outer horn $x : \Lambda^n_0 \to X$ is called *special* if $x|_{\Delta^1_{\{0,1\}}}$ is invertible.

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Theorem ([Joy02])

Let X be an ∞-category. Then every special outer horn $x : \Lambda^n_0 \to X$ can be filled.

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Corollary

An ∞-category X is Kan complex iff its fundamental category $\mathbf{cat}(X)$ is a groupoid.

Characterization of Kan complex

Theorem (Joyal)

Let X be a simplicial set, the following are equivalent:

1. *X is a Kan complex.*
2. *$X \to \mathbb{1}$ is right fibrant.*
3. *$X \to \mathbb{1}$ is left fibrant.*

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Proof(2) $\implies$ (1)

$X \to \mathbb{1}$ is right fibration $\implies$ $X \to \mathbb{1}$ is conservative $\implies$ $\mathbf{cat}(X) \to \mathbf{cat}(\mathbb{1})$ is conservative $\implies$ $\mathbf{cat}(X)$ is a groupoid $\implies$ X is Kan.

General setup

Setting

We denote our category $(\mathcal{C}, \mathcal{T})$, where $\mathcal{C}$ is an extensive category and $\mathcal{T}$ is a Grothendieck pretopology satisfying assumptions

- All finite products and finite coproducts exists in $\mathcal{C}$.
- All idempotents in $\mathcal{C}$ split.
- the pretopology $\mathcal{T}$ is retract stable.

With these assumptions, $(\mathcal{C}, \mathcal{T})$ is called a **Good geometric category**.

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Assumption $\star$[MZ15]

If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are maps such that $g \circ f$, and f is a cover, then g is a cover.

Examples of Good Geometric categories

The following are examples of Good Geometric categories which even satisfies assumption $\star$:

- $Sets$ with surjective submersions.
- Top with maps having global (local) continuous sections.
- Top with proper surjections.
- Top with open (or even et́ale) surjections.
- $Mfld_{fin}$ with surjective submersions.
- $Mfld_{Ban}$ with surjective submersions.
- $Mfld_{Hil}$ with surjective submersions.

Kan conditions

On a category $\mathcal{C}$, with Grothendieck pretopology $\mathcal{T}$, we say map between simplicial objects $X \to Y$ satisfies **Kan**(n, j)[or **Kan!**(n, j)] if the natural map below is a cover[or an isomorphism].

$$X_n = \hom(\Delta^n, X) \to \hom(\Lambda^n_j \to \Delta^n, X \to Y)$$

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$$X_n = \hom(\Delta^n, X) \to \hom(\Lambda^n_j \to \Delta^n, X \to Y)$$

- A simplicial object X is called left(right, or Kan) fibrant if the map $X \to \mathbb{1}$ satisfies Kan(n, j) for $0 \leq j < n$ $(0 < j \leq n, \text{or } 0 \leq j \leq n)$
- ∞-groupoids are Kan fibrant.

Groupoid Bibundle

Following the approach of Lurie, defining correspondence between simplicial sets, we define

Definition

Let G, H be two ∞- groupoids in $(\mathcal{C}, \mathcal{T})$. A $G - H$ bibundle is an inner Kan fibration $M \xrightarrow{\chi} \Delta^1$ with source $\chi^{-1}(0) = G$ and $\chi^{-1}(1) = H$.

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Remark

Since we are dealing with groupoids, we also need certain extra outer Kan conditions as defined by [Li15]. But with the added assumption on the cover, Li showed that these special outer horns can be automatically filled.

Right Kan implies Kan

Right Kan $\implies$ Inverse

Let X be a simplicial object in $(\mathcal{C}, \mathcal{T})$. Assume X is right Kan, that is $X \to \mathbf{1}$ is right Kan fibration. Then, Any arrow or 1−simplex $\in X_1$ has a left inverse, given by $Kan(2,2)$

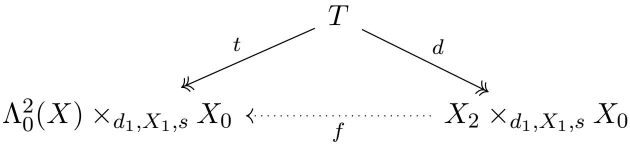
$$X_1 \xleftarrow{\sim} \Lambda^2_2(X) \times_{d_1, X_1, s} X_0 \twoheadleftarrow X_2 \times_{d_1, X_1, s} X_0$$

Right Kan $\implies$ Inverse

For the right inverse, consider the sequence of diagrams.

Right Kan $\implies$ Inverse

If we denote the higher simplex as T, then we have



Here $f \circ d = t$ is a cover, if we can show that d is a cover, then this implies f is a cover.

Right Kan $\implies$ Kan

Case 1

For a right Kan simplicial object X, the left outer horn Λ_0^2 can be filled.

Right Kan $\implies$ Kan

Lemma[Joy02]

We have

$$(\partial\Delta[m] \star \Delta[n]) \cup (\Delta[m] \star \partial\Delta[n]) = \partial\Delta[m+n+1]$$

$$(\Lambda^k[m] \star \Delta[n]) \cup (\Delta[m] \star \partial\Delta[n]) = \Lambda^k\Delta[m+n+1]$$

$$(\partial\Delta[m] \star \Delta[n]) \cup (\Delta[m] \star \Lambda^k[n]) = \Lambda^{m+1+k}[m+1+n]$$

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Lemma[Li15]

Let $f : A \to B$ be right (or boundary) collapsible extension and $g : X \to Y$ boundary (left collapsible) extension of simplicial sets. Then the induced inclusion is inner collapsible extension

$$A \star Y \bigcup_{A\star X} B \star X \to B \star Y.$$

Right Kan $\implies$ Kan

Proof

- Fill the degenerate simplex $\Delta^m\{0, 1^+, 2, \ldots, m\}$.
- When $m > 2$, Add another degenerate arrow $1^{++} \rightsquigarrow 1$.
- Fill $\Delta^2\{0, 1, 1^{++}\} \star \partial\Delta^{m-2}\{2, \ldots, m\}$ and also $\Delta^2\{0, 1^+, 1^{++}\} \star \partial\Delta^{m-2}\{2, \ldots, m\}$ by using degenerate simplices.
- Use $Kan(2, 2)$ on $\{1, 1^+, 1^{++}\}$ to join $1^+ \to 1$ and also to get $\Delta^2\{1, 1^+, 1^{++}\}$.
- Applying $Kan(3, 3)$ will give us $\Delta^3\{0, 1, 1^+, 1^{++}\}$.

Right Kan $\implies$ Kan

- Fill $\Delta^2\{1, 1^+, 1^{++}\} \star \partial\Delta^{m-2}\{2, \ldots, m\}$ and also $\Delta^3\{0, 1, 1^+, 1^{++}\} \star \partial\Delta^{m-2}\{2, \ldots, m\}$ using inner Kan conditions.
- Finally fill $\Lambda^m_1\{1, 1^+, 2 \ldots, m\}$ which will give the missing face $\{1, 2, \ldots, m\}$ and the inner horn $\Lambda^{m+1}_2\{0, 1, 1^+, 2, \ldots, m\}$ which will give us our simplex $\Delta^m\{0, 1, 2, \ldots, m\}$

Facing the Face maps.

Joyal's results

Definition

If $A \subseteq [n]$, the generalised horn $\Lambda^A[n]$ is the simplicial subset defined by $\Lambda^A[n] = \bigcup_{i \notin A} \partial_i \Delta[n]$.

Note that :

- $\Lambda^{\{k\}}[n] = \Lambda^k[n]$ for $k \in [0, n]$.
- $\Lambda^A[n] = d_0 \Delta^n$ for $A = [1, n]$.

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Proposition[Joy02]

Let $A \subseteq [n]$ be nonempty and $i_A : \Lambda^A[n] \subset \Delta[n]$ be the inclusion.

- if A is proper subset, then i_A is anodyne.
- if $A \subseteq [0, n-1]$, then i_A is left anodyne;
- if $A \subseteq [1, n]$, then i_A is right anodyne;
- if A^c is not an interval, then i_A is mid anodyne.

Face maps are covers (almost)

Let X be a right fibrant object, put $A = [1, n] \subseteq [n]$, then $i_A : \Lambda^A[n] = d_0(\Delta[n]) \to \Delta[n]$ is right anodyne, thus $d_0 : X_n \to X_{n-1}$ is a cover.

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Using the same techniques in the proof of previous proposition to work with topologically anodyne maps, we can further show that $d_i : X_n \to X_{n-1}$ is a cover for $i \neq n$ whenever X is a right fibrant object.

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WHAT ABOUT THE MAP $d_n : X_n \to X_{n-1}$?

Modified (better ?) assumption

Fact

If $g \circ f$ is a surjective submersion and f is a surjection, then g is a surjective submersion.

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Assumption $\star\star$

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps in a category $\mathcal{C}$ with a Grothendieck pretopology $\mathcal{T}$. If $g \circ f$ is a cover and f is an epimorphism, then g is a cover.

All the examples mentioned before satisfy assumption $\star\star$.

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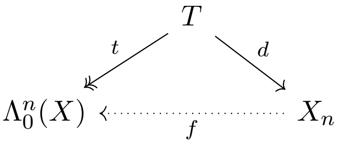
More Examples?

What about Fréchet manifolds ? Locally convex manifolds? Diffeological spaces?

Right Kan $\implies$ Kan

Recall that we had a higher simplex T, a cover and the face map



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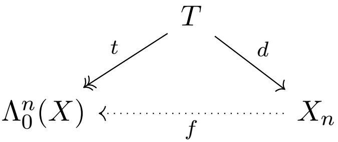

Here $f \circ d = t$ is a cover, d is surjective, hence an epimorphism, then this implies f is a cover.
And thus we Kan fill the outer horns !

Thank You!

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