Category Theory for Beginners

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"Category theory is generalized abstract nonsense." –Steenrod

"Category theory is quite the most general and abstract branch of pure mathematics. The corollary of a high degree of generality and abstraction is that the theory gives almost no assistance in solving the more specific problems within any of the subdisciplines to which it applies. It is a tool for the generalist, of little benefit for the practitioner." – Hoare

Some History

- Categories officially appeared in the paper "General theory of natural equivalences" by S. Eilenberg and S. Mac Lane (Trans. AMS 58, 1945, 231-294)
- Main applications were originally in algebraic topology and homology theory
- In 1960' Grothendieck et al. began using Category theory in algebraic geometry
- In 1970's Lawvere and others started applying Category theory in logic and related fields
- Later applications began appearing in computer science, linguistics, cognitive sciences, philosophy, quantum theory etc.

Let's Define things

Definition

A Category $\mathbf{C}$ consists of the following data :

- A collection $\mathcal{C}_0$ of objects.
- A collection $\mathcal{C}_1$ of morphisms.
- Maps $s, t : \mathcal{C}_1 \to \mathcal{C}_0$ called source and target map resp.
- A map $\mathbb{1}$: $\mathcal{C}_0 \to \mathcal{C}_1$ $(X \mapsto \mathbb{1}_X)$
- If $f, g \in \mathcal{C}_1$ such that $s(f) = t(g)$ then a morphism $f \circ g$ with $s(f \circ g) = s(g)$; $t(f \circ g) = t(f)$

Satisfying the given conditions:

1. Unit laws: for every $f : A \to B$, $f \circ \mathbb{1}_A = f = \mathbb{1}_B \circ f$
2. Assosiative law: for every $f : A \to B$, $g : B \to C$ and $h : C \to D$, $\quad h \circ (g \circ f) = (h \circ g) \circ f$

Category

- Let $\mathbf{C}$ be a category. For any two objects $A, B \in \mathbf{C}$ the collection of all morphism $A \to B$ is denoted by $Hom_C(A, B)$ or just $\mathbf{C}(A, B)$.
- Note that we defined category as the "collection" of objects and morphism, not as "sets".
- A category $\mathbf{C}$ where both $\mathcal{C}_0$ and $\mathcal{C}_1$ are sets is called a *small category*.
- A category where $\mathbf{C}(A, B)$ is a set for every object $A, B \in \mathbf{C}$ is called a *locally small category*.
- A subcategory $\mathbf{D}$ of $\mathbf{C}$ is such that $Obj(\mathbf{D}) \subseteq Obj(\mathbf{C})$ and $\mathbf{D}(A, B) \subseteq \mathbf{C}(A, B),\ \forall\ A, B \in Obj(\mathbf{D})$.

Examples

- $\mathbf{0}$ is the empty category with no object and no arrows.
- $\mathbf{1}$ is the category with just one object and a morphism which is the identity on object. $\overset{\circlearrowright}{1}$
- $\mathbf{2}$ is the category with two objects and one non-identity morphism between the two objects. $\overset{\circlearrowright}{1} \xrightarrow{i} \overset{\circlearrowright}{2}$
- Any pre-ordered set $(\mathbf{S}, \leq)$ forms a category, where the objects are members of $\mathbf{S}$ and morphisms are arrows pointing from x to y iff $x \leq y$.

More Examples

- **Set** is the category of sets, containing collection of sets as the objects and set maps as the morphisms.
- **Grp** is the category with objects as groups and morphisms as group homomorphism
- **Ring** is the category of rings and ring morphisms
- Similiarly **R-mod** and $\mathbf{Vect_k}$ are categories of R-modules and Vector fields over the field K respectively.
- **Top** is the category of topological spaces where **Top**(X,Y) is all the continous maps from X to Y. **hTop** is category of topological spaces with homotopy equivalent maps .
- A group itself can be considered as a category with a single object. $Obj(\mathbf{G}) = \{*\}$ and $Mor(\mathbf{G}) = G$

Monic and Epic

- An arrow f in $\mathbf{C}$ is called a *monomorphism*(monic) if for any $Z \in Obj(\mathbf{C})$ and two maps $g_1, g_2 \in \mathbf{C}(Z, X)$ such that

  $Z \underset{g_2}{\overset{g_1}{\rightrightarrows}} X \xrightarrow{f} Y$ satisfy $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$

- An arrow h in $\mathbf{C}$ is called an epic map or *epimorphism* if for any $Z \in Obj(\mathbf{C})$ and two maps $g_1, g_2 \in \mathbf{C}(Y, Z)$ such that

  $X \xrightarrow{h} Y \underset{g_2}{\overset{g_1}{\rightrightarrows}} Z$ satisfy $g_1 \circ h = g_2 \circ h$, then $g_1 = g_2$

Food for thought

Assuming objects in $\mathbf{C}$ are sets with some additional structure.

- Are all injective morphisms monomorphic ? What about the converse ?
- Are all surjective morphism epimorphic ? what about the converse ?

Isomorphisms and Groupoids

Definition

- An arrow $f \in \mathbf{C}(X, Y)$ is called an *isomorphism* if $\exists\ g$,
  $$X \underset{g}{\overset{f}{\rightleftarrows}} Y \quad \text{such that } f \circ g = \mathbb{1}_Y \ \&\ g \circ f = \mathbb{1}_X$$
- A category in which every morphism is an isomorphism is called a *Groupoid*.

- A group can be considered as a groupoid with one object.
- for a topological space X, the homotopy equivalent paths in X forms a groupoid πX called the **fundamental groupoid of X**

Food for thought

- f is both monic and epic $\iff$ f is an isomorphism ??

Functors

Definition

A (covariant) functor $F : \mathbf{C} \to \mathbf{D}$ consists of the following data :

- An object function $F : Obj(\mathbf{C}) \to Obj(\mathbf{D})$ which associates each object in $\mathbf{C}$ to an object in $\mathbf{D}$.
- A morphism function F, which associates for each map $X \xrightarrow{f} Y$ in $\mathbf{C}$, a map $F(X) \xrightarrow{F(f)} F(Y)$ in $\mathbf{D}$.

satisfying the following:

- $F(\mathbb{1}_X) = \mathbb{1}_{F(X)}, \ \forall\ X \in Obj(\mathbf{C})$
- $F(g \circ f) = F(g) \circ F(f)$ where g, f are composible arrows in $\mathbf{C}$

We can also have contravariant functors which reverse the direction of image of an arrow. that is

$X \xrightarrow{f} Y$ in $\mathbf{C}$, gets mapped to $F(X) \xleftarrow{F(f)} F(Y)$

Functors contd.

Let $F : \mathbf{C} \to \mathbf{D}$ be a functor (covariant) between locally small categories, then F is called

- a $faithful$ functor if $F : \mathbf{C}(A, B) \to \mathbf{D}(F(A), F(B))$ is injective for all $A, B \in \mathbf{C}$
- a $full$ functor if $F : \mathbf{C}(A, B) \to \mathbf{D}(F(A), F(B))$ is sujective for all $A, B \in \mathbf{C}$
- a $fully\ faithful$ functor if it is both full and faithful
- an *essentially surjective* functor if for each $D \in Obj(\mathbf{D}), \exists\ C \in Obj(\mathbf{C})$ such that $F(C) = D$.
- an *isomorphism* between categories is a functor $F : \mathbf{C} \to \mathbf{D}$ which is bijection both on objects and on arrows.

Food for thought

Does functor preserves monomorphisms ? then what about epimorphisms ? isomorphisms ?

Examples

- Identity functor $\mathbf{C} \to \mathbf{C}$, which sends objects and morphisms to itself.
- Constant functor $\mathbf{C} \to \mathbf{D}$ which sends all objects to a $D \in Obj(\mathbf{D})$ and all morphisms to $\mathbb{1}_D$.
- Forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ which sends each group to its underlying set and each group homomorphism to its underlying set map. (faithfull ?)
- $* : \mathbf{Vect_k} \to \mathbf{Vect_k}$, which associates each k-vector space V to its dual V*, is a contravariant functor.
- Let $\mathbf{C}$ be a locally small category, then each $A \in Obj(\mathbf{C})$ defines a covariant $hom\text{-}functor$ $h^A := Hom(A, -) : \mathbf{C} \to \mathbf{Set}$, which sends objects X in $\mathbf{C}$ to $Hom(A, X) = \mathbf{Set}(A, X)$. (on morphisms ? Is it a functor?)

the collection of all small categories and functors between them forms the category $\mathbf{Cat}$.

Natural transformations

Definition

Let $F, G : \mathbf{C} \to \mathbf{D}$ be functors, then a natural transformation η from F to G ($\eta : F \Rightarrow G$), is a collection of arrows $\{ \ F(A) \xrightarrow{\eta_A} G(A) \ , A \in Obj(\mathbf{C})\}$ in $\mathbf{D}$ such that the following diagram commutes for every $A \xrightarrow{f} B$ in $\mathbf{C}$

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\eta_B]{} & G(B) \end{array}$$

that is $G(f) \circ \eta_A = \eta_B \circ F(f)$.
If $\eta_A : F(A) \to G(A)$ is an isomorphims for every $A \in Obj(\mathbf{C})$, we call η to be a natural isomorphism.

An example

- Any vector space is naturally isomorphic to its double dual.

Consider the functors
$\mathbb{1}, D^2 : \mathbf{FDVect_k} \to \mathbf{FDVect_k}, \mathbb{1}(V) = V$ & $D^2(V) = V^{**}$
$V \xrightarrow{f} W \Rightarrow V^{**} \xrightarrow{f^{**}} W^{**}$. Define $\eta : \mathbb{1} \Rightarrow D^2$ where
$\eta_V : V \to V^{**}, x \mapsto L_x \in V^{**}$ such that $L_x(\hat{f}) = \hat{f}(x), \forall \hat{f} \in V^*$.
Check that the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{\eta_V} & V^{**} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{**}} \\ W & \xrightarrow[\eta_W]{} & W^{**} \end{array}$$

and show that η is a natural isomorphism.

Equivalence of Categories

Definition

Two categories $\mathbf{C}$ and $\mathbf{D}$ are said to be equivalent if there exists two functors $F : \mathbf{C} \to \mathbf{D}$ & $G : \mathbf{D} \to \mathbf{C}$ and two natural isomorphisms $\epsilon : GF \to \mathbb{1}_{\mathbf{C}}$ & $\eta : \mathbb{1}_{\mathbf{D}} \Rightarrow FG$. We say F (or G) is an equivalence between categories.

A Final Exercise

Prove that $F : \mathbf{C} \to \mathbf{D}$ is an equivalence of categories if and only if F is a fully faithful and essentially surjective functor.

References and suggested readings

- Mac Lane, Saunders (September 1998). Categories for the Working Mathematician. Graduate Texts in Mathematics. 5 (Second ed.). Springer. ISBN 0-387-98403-8. Zbl 0906.18001.
- Kashiwara, M., Schapira, P. (2006). Categories and Sheaves. In Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg. https://doi.org/10.1007/3-540-27950-4
- The nLab https://ncatlab.org/nlab/show/HomePage

Thank You!