

Some Concepts in Category Theory

- CMIT Lecture

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Let's Recall

- CAT, FUN, NAT
- Mono, epi & isomorphisms
- Initial, terminal & zero objects
- Yoneda Lemma

o Universal Property of Kernel

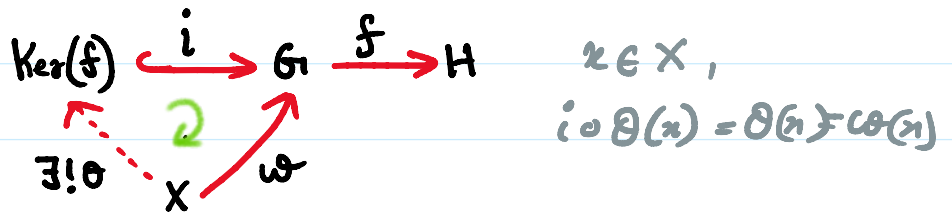
Let $f: G \rightarrow H$ be a gp homomorphism. Then by defn.
 $\text{Ker}(f) := \{g \in G \mid f(g) = 0\}$

Note that $\text{Ker}(f) \subset G$, so we have the inclusions
 $\text{Ker}(f) \xrightarrow{i} G \xrightarrow{f} H \quad \& \quad f \circ i = 0.$

Now Assume X , a group & $\omega: X \rightarrow G$ such that
 $f \circ \omega = 0$

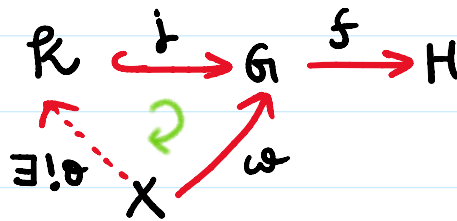
$\Rightarrow \forall x \in X, f \circ \omega(x) = 0 \Rightarrow \omega(x) \in \text{Ker}(f)$

so we have a unique map $\Theta: X \rightarrow \text{Ker}(f)$
 $x \mapsto \omega(x)$



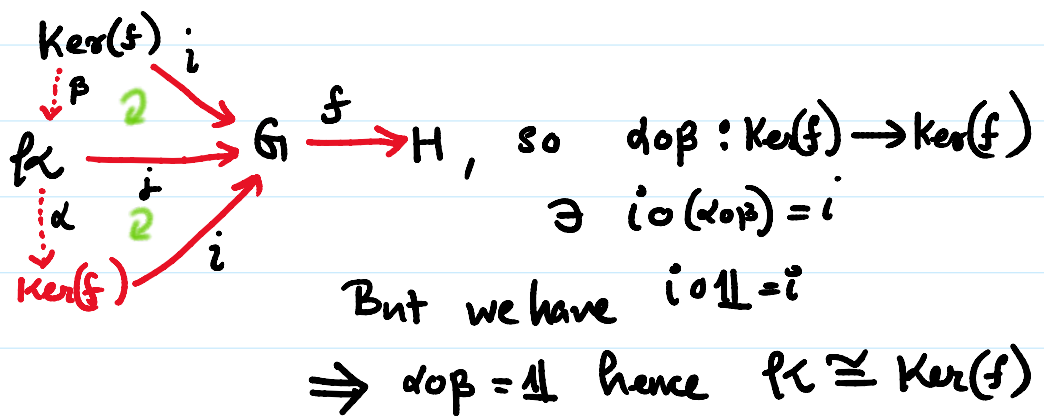
Is Kernel, the only gp with this property??

Assume, \exists another gp $K \xrightarrow{i} G \xrightarrow{f} H$ with $f \circ i = 0$
 & for any $\omega: X \rightarrow G$



since $f \circ i = 0$, $\exists! \alpha: K \rightarrow \text{Ker}(f)$
 also $f \circ i = 0 \Rightarrow \exists! \beta: \text{Ker}(f) \rightarrow K$

Why?



★ Universal property:

Given a group homomorphism $f: G \rightarrow H$,
 $(\text{Ker}(f), i)$ is the unique group with $\text{Ker}(f) \xrightarrow{i} G \xrightarrow{f} H$
 $f \circ i = 0$ such that for any other (X, ω) $\omega: X \rightarrow G$
 with $f \circ \omega = 0$, $\exists! \theta: X \rightarrow \text{Ker}(f)$ $\exists \omega = i \circ \theta$

• Cokernel

Let $f: V \rightarrow W$ be a linear transformation between vector spaces V, W .

$$\text{Coker}(f) := W / \text{Im}(f).$$

We have

$$V \xrightarrow{f} W \xrightarrow{p} \text{Coker}(f) \quad p \circ f = 0$$

with the property if $\exists \bar{p}: W \rightarrow Y$, Y being a v.s. with $\bar{p} \circ f = 0$, then $\exists! \theta: \text{Coker}(f) \rightarrow Y$

$$\begin{array}{ccccc} V & \xrightarrow{f} & W & \xrightarrow{p} & \text{Coker}(f) \\ & & \searrow \bar{p} & \curvearrowright & \swarrow \exists! \theta \\ & & X & & \end{array}$$

EXERCISE :: State the Universal Property of Cokernel

- What might be Cokernel of a map $f: G \rightarrow H$ in category of groups?

• Equalizer

Let $f, g: A \rightarrow B$ be two set maps.

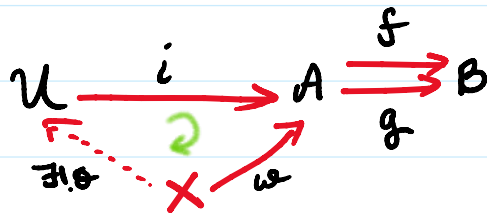
Define $U = \{a \in A \mid f(a) = g(a)\}$, then we have

$$U \xrightarrow{i} A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B \quad \text{with } f \circ i = g \circ i$$

if $\exists x \xrightarrow{w} A$, with $f \circ w = g \circ w$. Then $\forall x \in X$

$$f \circ w(x) = g \circ w(x) \Rightarrow w(x) \in U$$

so we have a unique map $\theta: X \rightarrow U$



The pair (U, i) is called the equalizer for $f, g: A \rightarrow B$

Exercise: Is it Unique?

State the Universal property of Equalizer.

In Set, equalizer \Leftrightarrow monomorphism \Leftrightarrow injective

Remark

- Kernel is a special case of an equalizer

$$\text{Ker}(G \xrightarrow{f} H) = \text{Equalizer} \left(G \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{0} \end{array} H \right)$$

- In category of groups/vector space etc

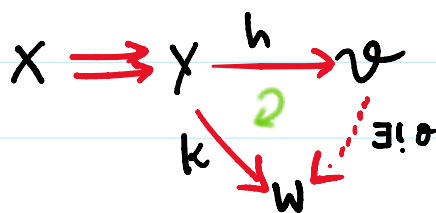
$$\text{Equalizer} \left(G \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} H \right) = \text{Ker}(f - g)$$

- Coequalizer

Given morphisms $X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y$, the coequalizer is the universal object

$$X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y \xrightarrow{h} Z \quad h \circ f = h \circ g$$

with the property, if $\exists Y \xrightarrow{k} W$ with $k \circ f = k \circ g$



Exercise: What is a coequalizer in Set? Grp?

- Products and Coproducts of two objects.

Recall that if A, B are sets, $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

Do we have a universal property?

Of course. As soon as we define $A \times B$, we note that we get two maps (projections)

$$p_1: A \times B \longrightarrow A \\ (a, b) \longmapsto a$$

$$p_2: A \times B \longrightarrow B \\ (a, b) \longmapsto b$$

Suppose given a set X & two maps q_1, q_2

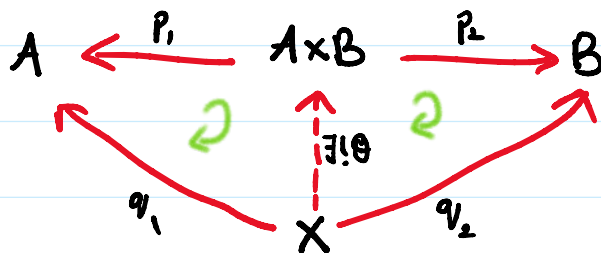
$$q_1: X \longrightarrow A$$

$$q_2: X \longrightarrow B$$

then, we can define a map (unique?)

$$\theta: X \longrightarrow A \times B$$

$$x \longmapsto (q_1(x), q_2(x))$$



Voila!!!

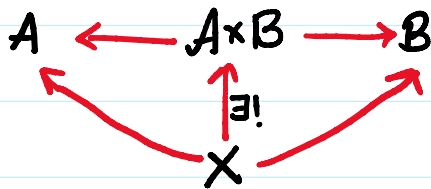
if it exists

Given two objects A, B in a category, the product of A & B denoted as $A \times B$ is the unique object with maps

Given two objects A, B in a category, the product of A & B denoted as $A \times B$ is the unique object with maps

$$A \xleftarrow{p_1} A \times B \xrightarrow{p_2} B$$

such that given any other

$$A \xleftarrow{q_1} X \xrightarrow{q_2} B, \exists! \theta : X \rightarrow A \times B$$


Exercise: Prove that product of A & B is unique if it exists

- In category of groups, direct product is the product
- In category of topological spaces, the product is the cartesian product with product topology.

Interesting example

Let (A, \leq) be a partially ordered set. That is

- \leq is reflexive
- \leq is antisymmetric
- \leq is transitive

We can consider (A, \leq) as a category

$$\text{Obj}(A) = A$$

$$A \rightarrow B \text{ exists iff } A \leq B.$$

What is the product of A & B .

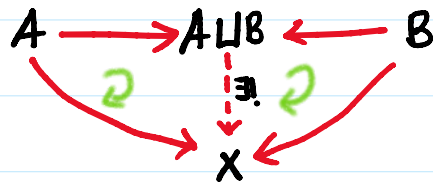
What about in a category \mathbb{Z} , where

$$\text{Obj}(\mathbb{Z}) = \mathbb{Z}^+$$

$$m \rightarrow n \text{ exists iff } m|n.$$

$\exists f: m \rightarrow n$ exists iff $m|n$.

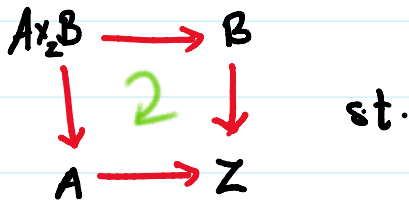
Coproducts



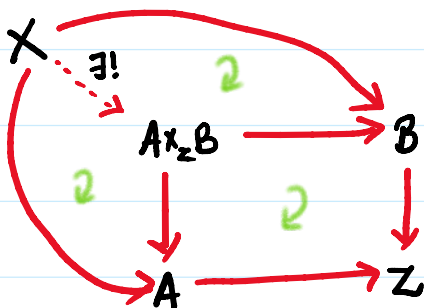
- State the universal property.
- Coproduct of two sets A, B is their disjoint union.
- coproduct of two vector spaces is the direct sum.
- Coproduct in (A, \leq) , the poset? ↖ convince yourself

Pull backs

Given $A \rightarrow Z$ and $B \rightarrow Z$, pull back is

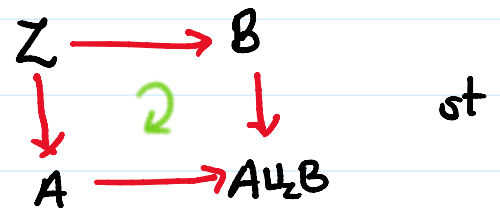


if $\exists X \rightarrow B, X \rightarrow A$ with

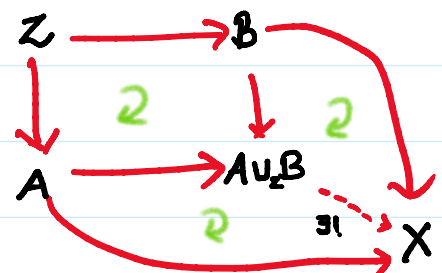


Push forwards

Given $Z \rightarrow B$ and $A \rightarrow B$, push fwd is



if $\exists A \rightarrow X, B \rightarrow X$ commuting



UNIVERSAL ARROWS

Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor, given $A \in \text{Obj}(\mathcal{D})$
 Define a category \mathcal{C}_A

$$\text{Obj}(\mathcal{C}_A) = \{ (C, f) \mid F(C) \xrightarrow{f} A \}$$

$$\text{Hom}((C, f), (D, g)) := \left\{ C \xrightarrow{\alpha} D \mid \begin{array}{ccc} F(C) & \xrightarrow{F(\alpha)} & F(D) \\ f \searrow & \text{\scriptsize 2} & \swarrow g \\ & A & \end{array} \right\}$$

A **Universal Arrow** from F to A is a terminal object in \mathcal{C}_A

that is, it is a pair $(V, \theta) \in \text{Obj}(\mathcal{C}_A)$ with the property, if $\exists f: F(C) \rightarrow A$ for any $C \in \mathcal{C}$ then $\exists!$ map

$$\begin{array}{ccc} F(C) & \xrightarrow{\quad} & F(V) \\ f \searrow & \text{\scriptsize 2} & \swarrow \theta \\ & A & \end{array}$$

We can also consider \mathcal{C}^A

$$\text{Obj}(\mathcal{C}^A) = \{ (C, f) \mid A \xrightarrow{f} F(C) \}$$

$$\text{Hom}((C, f), (D, g)) = \left\{ C \xrightarrow{\alpha} D \mid \begin{array}{ccc} & A & \\ f \swarrow & \text{\scriptsize 2} & \searrow g \\ F(C) & \xrightarrow{F(\alpha)} & F(D) \end{array} \right\}$$

A **Universal Arrow** from A to F is an initial object in \mathcal{C}^A (Work out what this means)

initial object in \mathcal{C} (Work out what that means)

LIMITS & COLIMITS

Let J be a small category. Let $F: J \rightarrow \mathcal{C}$.
 F belongs to the functor category $\text{Fun}[J, \mathcal{C}]$.

We have the functor

$$\Delta: \mathcal{C} \rightarrow \text{Fun}[J, \mathcal{C}]$$

$$A \mapsto \Delta_A$$

$$\Delta_A(j \rightarrow j') = A \xrightarrow{1_A} A$$

then a **limit of shape J** in \mathcal{C} is a universal arrow from Δ to F .

We use $\varprojlim F$ to denote the limit of F .

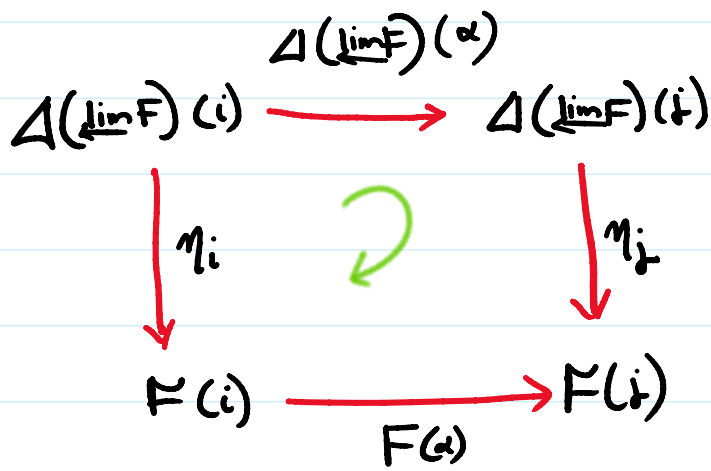
A **colimit** of F of shape J is a universal arrow from F to Δ (denoted $\varinjlim F$)

Okay!!! But what does it mean,???

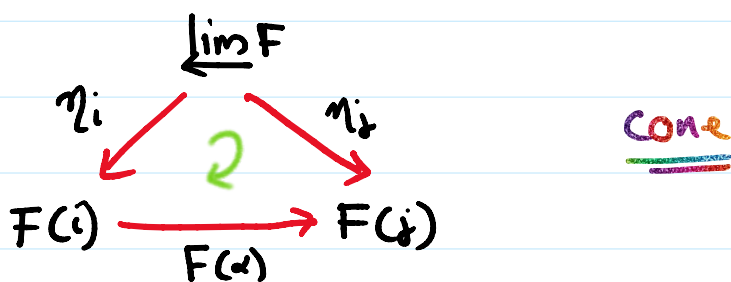
Universal arrow from Δ to $F \Rightarrow$ terminal object in $\text{Fun}[J, \mathcal{C}]_F$

which is a pair $(\varprojlim F, \eta)$ $\Delta(\varprojlim F) \xrightarrow{\eta} F$

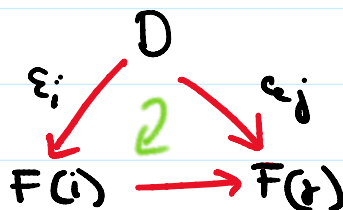
So given $(i \xrightarrow{\alpha} j)$ in J



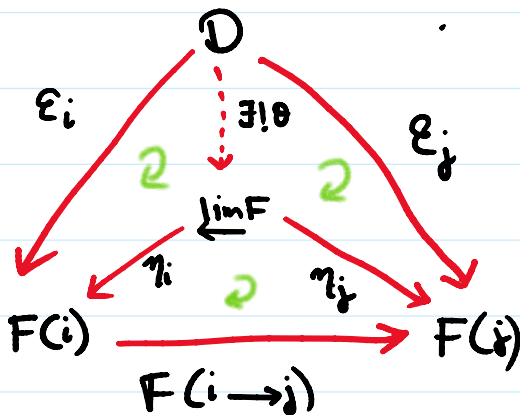
\Downarrow



Given any other $(D, \epsilon) \in \text{Cone}[\mathcal{J}, \mathcal{B}]_F$
is a cone

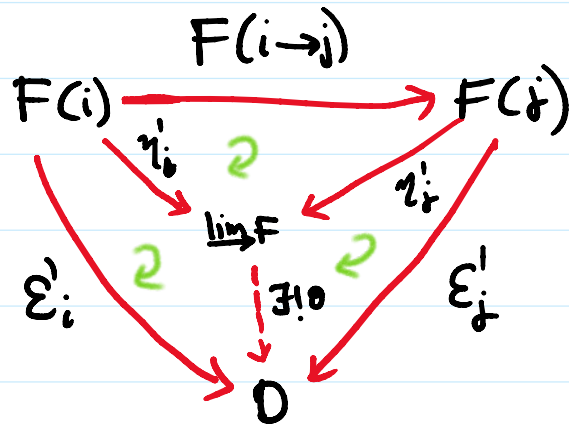


then $\exists!$ map $\theta: D \rightarrow \varinjlim F$



Colimit

Colimit



THANKS

Oh! Wait. There's more (provided we have time)

We have seen that given a cone (D, ϵ) with base F , there exist a unique map $D \xrightarrow{\theta} \varprojlim F$

Conversely if given an arrow $D \xrightarrow{\theta} \varprojlim F$, $(D, \gamma_0 \theta)$ forms a cone. Thus we have the bijection

$$\text{Cone}(-, F) \cong \text{Hom}(-, \varprojlim F)$$

This gives another characterization of a limit.

$\varprojlim F$ is the object representing the functor $\text{Cone}(-, F)$
 $\text{Cone}(-, F) : \mathcal{C}^{\text{op}} \rightarrow \text{Set}$