Some Concepts in Category Theory

- CMIT Lecture — Kalin krishna

Let's Recall

- CAT, FUN, NAT
- Mono, epi & isomorphisms
- Initial, terminal & zero objects
- Yoneda Lemma

◦ Universal Property of Kernel

Let $f: G \to H$ be a gp homomorphism. Then by defn.

$$\operatorname{Ker}(f) := \{ g \in G \mid f(g) = 0 \}$$

Note that $\operatorname{Ker}(f) \subset G$, so we have the inclusion

$$\operatorname{Ker}(f) \xhookrightarrow{i} G \xrightarrow{f} H \quad \& \quad f \circ i = 0.$$

Now Assume X, a group & $\omega : X \to G$ such that $f \circ \omega = 0$

$\Rightarrow \forall x \in X$, $f \circ \omega(x) = 0 \Rightarrow \omega(x) \in \operatorname{Ker} f$

so we have a unique map $\theta : X \to \operatorname{Ker}(f)$

$$x \mapsto \omega(x)$$

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, hook, "i"] & G \arrow[r, "f"] & H \\ & X \arrow[ul, dashed, "\exists!\theta"] \arrow[u, "\omega"'] \arrow[u, phantom, shift right=6, "\circlearrowleft"] & \end{tikzcd}$$

$x \in X$,
$i \circ \theta(x) = \theta(x) = \omega(x)$

$\boxed{\text{Is Kernel, the only gp with this property??}}$

Assume, $\exists$ another gp $K \xrightarrow{j} G \xrightarrow{f} H$ with $f \circ j = 0$
& for any $\omega : X \to G$

$$\begin{tikzcd} K \arrow[r, hook, "j"] & G \arrow[r, "f"] & H \\ & X \arrow[ul, dashed, "\exists!\theta"] \arrow[u, "\omega"'] \arrow[u, phantom, shift right=6, "\circlearrowleft"] & \end{tikzcd}$$

since $f \circ j = 0$, $\exists!\ \alpha : K \to \mathrm{Ker}(f)$ 　 Why?
also $f \circ i = 0 \Rightarrow \exists!\ \beta : \mathrm{Ker}(f) \longrightarrow K$

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[d, dashed, "\beta"] \arrow[dr, "i"] & & \\ K \arrow[r, "j"] \arrow[d, dashed, "\alpha"] & G \arrow[r, "f"] & H \\ \mathrm{Ker}(f) \arrow[ur, "i"'] & & \end{tikzcd}$$

, so $\alpha \circ \beta : \mathrm{Ker}(f) \to \mathrm{Ker}(f)$
$\ni i \circ (\alpha \circ \beta) = i$

But we have $i \circ \mathbb{1} = i$

$\Rightarrow \alpha \circ \beta = \mathbb{1}$ hence $K \cong \mathrm{Ker}(f)$

★ <u>Universal property</u>:

Given a group homomorphism $f : G \to H$,
$(\mathrm{Ker}(f), i)$ is the unique group with $\mathrm{Ker}(f) \xrightarrow{i} G \xrightarrow{f} H$
$f \circ i = 0$ such that for any other (X, ω) $\omega : X \to G$
with $f \circ \omega = 0$, $\exists!\ \theta : X \to \mathrm{Ker}(f)$ $\ni \omega = i \circ \theta$

- <u>Cokernel</u>

Let $f: V \to W$ be a linear transformation between vector spaces V, W.

$$\mathrm{Coker}(f) := W/\mathrm{Im}(f).$$

We have

$$V \xrightarrow{f} W \xrightarrow{p} \mathrm{Coker}(f) \qquad p \circ f = 0$$

with the property if $\exists\ \bar{p}: W \to Y$, Y being a v.s with $\bar{p} \circ f = 0$, then $\exists!\ \theta: \mathrm{Coker}(f) \to Y$

$$\begin{tikzcd} V \arrow[r, "f"] & W \arrow[r, "p"] \arrow[dr, "\bar{p}"'] & \mathrm{Coker}(f) \arrow[d, dashed, "\exists!\theta"] \\ & & X \end{tikzcd}$$

EXERCISE: State the Universal Property of Cokernel

- What might be Cokernel of a map $f: G \to H$ in category of groups?

- <u>Equalizer</u>

Let $f, g: A \to B$ be two set maps.
Define $\mathcal{U} = \{a \in A \mid f(a) = g(a)\}$, then we have

$$\mathcal{U} \xrightarrow{i} A \underset{g}{\overset{f}{\rightrightarrows}} B \quad \text{with} \quad f \circ i = g \circ i$$

if $\exists\ X \xrightarrow{\omega} A$, with $f \circ \omega = g \circ \omega$. Then $\forall x \in X$

$$f \circ \omega(x) = g \circ \omega(x) \Rightarrow \omega(x) \in \mathcal{U}$$

so we have a unique map $\theta: X \to \mathcal{U}$

$$\begin{tikzcd}
U \arrow[r, "i"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
& X \arrow[ul, dashed, "\exists!\theta"] \arrow[u, "w"'] \arrow[ul, phantom, "\circlearrowleft"]
\end{tikzcd}$$

The pair (U,i) is called the equaliser for $f,g: A \to B$

Exercise: Is it Unique?

State the Universal property of Equalizer.

In Set, equalizer $\Leftrightarrow$ monomorphism $\Leftrightarrow$ injective

Remark

- Kernel is a special case of an equalizer
  $\mathrm{Ker}(G \xrightarrow{f} H) = \mathrm{Equalizer}\,(G \overset{f}{\underset{0}{\rightrightarrows}} H)$.

- In category of groups / vector space etc
  $\mathrm{Equalizer}(G \overset{f}{\underset{g}{\rightrightarrows}} H) = \mathrm{Ker}\,(f-g)$.

- Coequalizer

Given morphisms $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, the coequalizer is the universal object

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y \xrightarrow{h} V \qquad h\circ f = h\circ g$$

with the property, if $\exists\ Y \xrightarrow{k} W$ with $k\circ f = k\circ g$

$$\begin{tikzcd}
X \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[r, "h"] \arrow[dr, "k"'] \arrow[dr, phantom, "\circlearrowleft", shift left=3] & V \arrow[d, dashed, "\exists!\theta"] \\
& & W
\end{tikzcd}$$

Exercise: What is a coequalizer in Set? Grp?

• **Products and Coproducts of two objects.**

Recall that if A, B are sets, $A \times B = \{(a,b) \mid a \in A, b \in B\}$.

$\boxed{\text{Do we have a universal property?}}$

Of course. As soon as we define $A \times B$, we note that we get two maps (projections)

$$p_1 : A \times B \longrightarrow A \qquad\qquad p_2 : A \times B \longrightarrow B$$
$$(a,b) \longmapsto a \qquad\qquad\qquad (a,b) \longmapsto b$$

Suppose given a set X & two maps q_1, q_2

$$q_1 : X \longrightarrow A \qquad\qquad q_2 : X \longrightarrow B$$

then, we can define a map (unique?)

$$\theta : X \longrightarrow A \times B$$
$$x \longmapsto (q_1(x), q_2(x))$$

$$\begin{tikzcd}
A & A \times B \arrow[l, "p_1"'] \arrow[r, "p_2"] & B \\
& X \arrow[ul, "q_1"] \arrow[u, dashed, "\exists! \theta"'] \arrow[ur, "q_2"'] \arrow[ul, phantom, "\circlearrowleft", shift left=4] \arrow[ur, phantom, "\circlearrowleft", shift right=4] &
\end{tikzcd}$$

Voila!!!

if it exists

Given two objects A, B in a category, the product of A & B denoted as $A \times B$ is the unique object with maps

Given two objects A, B in a category, the product of A & B denoted as $A \times B$ is the unique object with maps

$$A \xleftarrow{p_1} A\times B \xrightarrow{p_2}$$ such that given any other $A \xleftarrow{\alpha_1} X \xrightarrow{\alpha_2} B$, $\exists!\ \theta : X \longrightarrow A\times B$

$$\begin{tikzcd}
A & A\times B \arrow[l] \arrow[r] & B \\
 & X \arrow[ul] \arrow[u, "\exists!"'] \arrow[ur] &
\end{tikzcd}$$

<u>Exercise</u> : Prove that product of A & B is unique if it exists

- In category of groups, direct product is the product
- In category of topological spaces, the product is the cartesian product with product topology.

<u>Interesting example</u>

Let $(A, \leq)$ be a partially ordered set. That is
- $\leq$ is reflexive
- $\leq$ is antisymmetric
- $\leq$ is transitive

we can consider $(A, \leq)$ as a category

$Obj(A) = A$

$A \longrightarrow B$ exists iff $A \leq B$.

What is the product of A & B.

What about in a category $\mathcal{Z}$, where

$Obj(\mathcal{Z}) = \mathbb{Z}^+$

$m \longrightarrow n$ exists iff $m \mid n$.

$Obj(\mathbb{Z}) = \mathbb{Z}$

$m \longrightarrow n$ exists iff $m | n$.

Coproducts

$$\begin{tikzcd}
A \arrow[r] \arrow[dr, bend right] \arrow[dr, phantom, "\circlearrowleft" near start] & A \sqcup B \arrow[d, dashed, "\exists!"] & B \arrow[l] \arrow[dl, bend left] \arrow[dl, phantom, "\circlearrowleft" near start] \\
& X &
\end{tikzcd}$$

- State the universal property.
- Coproduct of two sets A, B is their disjoint union. Convince yourself
- coproduct of two vector spaces is the direct sum.
- Coproduct in $(A, \leq)$, the poset ?

Pull backs & Push forwards

Pull backs

Given $\begin{tikzcd} & B \arrow[d] \\ A \arrow[r] & Z \end{tikzcd}$, pull back is

$$\begin{tikzcd}
A \times_Z B \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d] \\
A \arrow[r] & Z
\end{tikzcd}$$

st. if $\exists$ $X \to B$, $X \to A$ with

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& A \times_Z B \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d] \\
& A \arrow[r] & Z
\end{tikzcd}$$

Push forwards

Given $\begin{tikzcd} Z \arrow[r] \arrow[d] & B \\ A & \end{tikzcd}$, push fwd is

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d] \\
A \arrow[r] & A \sqcup_Z B
\end{tikzcd}$$

st if $\exists$ $A \to X$, $B \to X$ commuting

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d] \arrow[ddr, bend left] & \\
A \arrow[r] \arrow[drr, bend right] & A \sqcup_Z B \arrow[dr, dashed, "\exists!"] & \\
& & X
\end{tikzcd}$$

Universal Arrows

Let $F: \mathcal{C} \to \mathcal{D}$ be a functor, given $A \in \mathrm{Obj}(\mathcal{D})$

Define a category $\mathcal{C}_A$

$$\mathrm{Obj}(\mathcal{C}_A) = \{ (C, f) \mid F(C) \xrightarrow{f} A \}$$

$$\mathrm{Hom}\big((C,f),(D,g)\big) := \left\{ C \xrightarrow{\alpha} D \;\middle|\; \begin{tikzcd} F(C) \arrow[rr, "F(\alpha)"] \arrow[dr, "f"'] & \arrow[d, phantom, "\circlearrowleft"] & F(D) \arrow[dl, "g"] \\ & A & \end{tikzcd} \right\}$$

A **Universal Arrow** from F to A is a terminal object in $\mathcal{C}_A$

that is, it is a pair $(V, \theta) \ni F(V) \xrightarrow{\theta} A$
with the property, if $\exists\ f: F(C) \to A$ for any $C \in \mathcal{C}$
then $\exists!$ map

$$\begin{tikzcd} F(C) \arrow[rr] \arrow[dr, "f"'] & \arrow[d, phantom, "\circlearrowleft"] & F(V) \arrow[dl, "\theta"] \\ & A & \end{tikzcd}$$

We can also consider $\mathcal{C}^A$

$$\mathrm{Obj}(\mathcal{C}^A) = \{ (C, f) \mid A \xrightarrow{f} F(C) \}$$

$$\mathrm{Hom}\big((C,f),(D,g)\big) = \left\{ C \xrightarrow{\alpha} D \;\middle|\; \begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\circlearrowleft"] & \\ F(C) \arrow[rr, "F(\alpha)"'] & {} & F(D) \end{tikzcd} \right\}$$

A **Universal Arrow** from A to F is an initial object in $\mathcal{C}^A$ (Work out what that means) initial object in $\mathscr{C}$.. (Work out what that means)

Limits & Colimits

Let J be a small category. Let $F : J \to \mathscr{C}$.
F belongs to the functor category $\mathrm{Fun}[J, \mathscr{C}]$.
We have the functor

$$\Delta : \mathscr{C} \longrightarrow \mathrm{Fun}[J, \mathscr{C}]$$
$$A \longmapsto \Delta_A \qquad \Delta_A(j \longrightarrow j') = A \xrightarrow{1_A} A$$

then a limit of shape J in $\mathscr{C}$ is a universal arrow from Δ to F.
We use $\varprojlim F$ to denote the limit of F.

A colimit of F of shape J is an universal arrow from F to Δ (denoted $\varinjlim F$)

Okay!!! But what does it mean???

Universal arrow from Δ to $F \Rightarrow$ terminal object in $\mathrm{Fun}[J, \mathscr{C}]_F$

which is a pair $(\varprojlim F, \eta)$ $\quad \Delta(\varprojlim F) \overset{\eta}{\Longrightarrow} F$

So given $(i \xrightarrow{\alpha} j)$ in J

$$\begin{tikzcd}
\Delta(\varprojlim F)(i) \arrow[r, "\Delta(\varprojlim F)(\alpha)"] \arrow[d, "\eta_i"'] \arrow[dr, phantom, "\circlearrowleft"] & \Delta(\varprojlim F)(j) \arrow[d, "\eta_j"] \\
F(i) \arrow[r, "F(\alpha)"'] & F(j)
\end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd}
& \varprojlim F \arrow[dl, "\eta_i"'] \arrow[dr, "\eta_j"] \arrow[d, phantom, "\circlearrowleft"] & \\
F(i) \arrow[rr, "F(\alpha)"'] & {} & F(j)
\end{tikzcd}$$

cone

Given any other $(D, \varepsilon) \in \mathrm{Fun}[\mathcal{J}, \mathcal{C}]_F$

ie a cone

$$\begin{tikzcd}
& D \arrow[dl, "\varepsilon_i"'] \arrow[dr, "\varepsilon_j"] \arrow[d, phantom, "\circlearrowleft"] & \\
F(i) \arrow[rr] & {} & F(j)
\end{tikzcd}$$

then $\exists!$ map $\theta : D \to \varprojlim F$

$$\begin{tikzcd}
& D \arrow[ddl, bend right, "\varepsilon_i"'] \arrow[ddr, bend left, "\varepsilon_j"] \arrow[d, dashed, "\exists!\theta"] & \\
& \varprojlim F \arrow[dl, "\eta_i"'] \arrow[dr, "\eta_j"] \arrow[d, phantom, "\circlearrowleft"] & \\
F(i) \arrow[rr, "F(i \to j)"'] & {} & F(j)
\end{tikzcd}$$

Colimit

Colimit

$$\begin{tikzcd}
F(i) \arrow[rr, "F(i\to j)"] \arrow[dr, "\eta'_i"] \arrow[ddr, bend right, "\varepsilon'_i"'] & & F(j) \arrow[dl, "\eta'_j"'] \arrow[ddl, bend left, "\varepsilon'_j"] \\
& \underrightarrow{\lim} F \arrow[d, dashed, "\exists ! \theta"] & \\
& D &
\end{tikzcd}$$

THANKS

Oh! Wait. There's more (provided we have time)

We have seen that given a cone (D, ε) with base F, there exist a unique map $D \xrightarrow{\theta} \underleftarrow{\lim} F$

Conversely if given an arrow $D \xrightarrow{\theta} \underleftarrow{\lim} F$, $(D, \eta \circ \theta)$ forms a cone. Thus we have the bijection

$$\mathrm{Cone}(-, F) \cong \mathrm{Hom}(-, \underleftarrow{\lim} F)$$

This gives another characterization of a limit.

$\underleftarrow{\lim} F$ is the object representing the functor $\mathrm{Cone}(-, F)$

$$\mathrm{Cone}(-, F) : \mathcal{C}^{op} \to \mathrm{Set}$$