Higher groupoids, actions and bibundles

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Outline

Overview

Background

- Higher groupoids play a crucial role in the active research in the interplay between higher category theory and different other fields within mathematics.
- Higher Lie groupoids acts as charts for differentiable stacks.
- [RZ20] built a higher category where these higher Lie groupoids can live in, and the morphisms are zig-zag morphisms with weak equivalence conditions imposed.
- [Li15] worked out some details of 2-groupoid actions, bibundles, and principal bundles, and gave correspondence to actions of categorified groupoids.

Objectives

1. To study the work of [Li15] on higher groupoid actions, bibundles, and principal bundles, and generalize these notions and work out the details for an n- groupoids in general higher category.
2. To explore the works in [RZ20] and to construct a higher category to host the general higher n-groupoids.

Introduction

Category

Definition

A Category $\mathbf{C}$ consists of the following data:

- A collection $\mathcal{C}_0$ of objects.
- A collection $\mathcal{C}_1$ of morphisms.
- Maps $s, t : \mathcal{C}_1 \to \mathrm{C}_0$ called source and target map resp.
- A map $\mathbb{1}$: $\mathcal{C}_0 \to \mathrm{C}_1$ $(X \mapsto \mathbb{1}_X)$
- If $f, g \in \mathcal{C}_1$ such that $s(f) = t(g)$ then a morphism $f \circ g$ with $s(f \circ g) = s(g)$; $t(f \circ g) = t(f)$

Satisfying the given conditions:

1. Unit laws: for every $f : A \to B$, $f \circ \mathbb{1}_A = f = \mathbb{1}_B \circ f$
2. Assosiative law: for every $f : A \to B$, $g : B \to C$ and $h : C \to D$, $\quad h \circ (g \circ f) = (h \circ g) \circ f$

Groupoids

Definition

A groupoid is a category where all the morphisms are invertible.

Examples

- A group G is a groupoid with a single object.
- Any equivalence relation is a groupoid.
- Fundamental groupoid of a topological space.

Lie groupoids

Definition

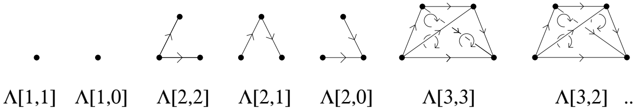
A Lie groupoid is a groupoid where both G_0 and G_1 are smooth manifolds, and all the structure maps (source, target, identity, inverse, composition) are smooth. We also require the source and target maps $s, t : G_1 \rightrightarrows G_0$ to be submersions.

Simplicial picture

Notations

Let Δ^n denote the simplicial n-simplex and the horn Λ^n_k ($\Lambda[n,k]$) is the simplicial subset of Δ^n defined by

$$[m] \mapsto \{f \in \hom([m],[n]) \mid [n] - [k] \not\subseteq f[m]\}$$



Remark

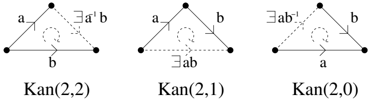
If you consider the points as objects and the arrows as morphisms in a category, we can see that $\Lambda[2,1]$ gives a pair of composable maps.

Kan conditions

Motivated by the above remark, we make the following definition

Definition

We say the simplicial set X satisfy $Kan(m, j)$ if any map from the horn Λ_j^m to X extends to a map from Δ^m to X. A **Kan simplicial** set is a simplicial set X which satisfies $Kan(m, j)$ for all m and $j \leq m$.



If the extension associated to $Kan(m, j)$ is unique, then we denote it as $Kan!(m, j)$.

n-groupoids

n-groupoids

Definition

An *n-groupoid* X is a simplicial set which satisfies $Kan(m, j)$ for all $0 \leq j \leq m$, $m \geq 1$ and also satisfies $Kan!(m, j)$ for all $0 \leq j \leq m$, $m > n$

- The nerve of a groupoid is a 1- groupoid.

Singleton Grothendieck Pretopolgy

A collection of morphisms $\mathcal{T}$ is called a singleton Grothendieck pretopolgy in $\mathcal{C}$ if

- every isomorphisms in $\mathcal{C}$ is in $\mathcal{T}$.
- the pullback of every morphism in $\mathcal{T}$ exists and is in $\mathcal{T}$.
- if $f, g \in \mathcal{T}$ are composable, then $f \circ g \in \mathcal{T}$.

Morphisms in $\mathcal{T}$ are called covers in $\mathcal{C}$.

Examples

- Category of smooth manifolds with surjective submersions.
- Category of topological spaces with surjective etale maps.

modified Kan conditions

In the setup of category with a singleton grothendieck pretopology, we modify our Kan conditions as given:

$Kan(m, j)$: The map $\hom(\Delta m, X) \to \hom(\Lambda^m_j, X)$ is a cover in $(\mathcal{C}, \mathcal{T})$.
$Kan!(m, j)$: The map $\hom(\Delta m, X) \to \hom(\Lambda^m_j, X)$ is an isomorphism in $\mathcal{C}$.

Replacing the horn Λ^m_j with the boundary $\partial\Delta^m$, we obtain the Acyclic conditions $Acyc(m)$.

n-groupoids in $(\mathcal{C}, \mathcal{T})$

Definition

An n-groupoid object X $(n \in \mathbb{N} \cup \infty)$ is a simplicial object in $(\mathcal{C}, \mathcal{T})$ which satisfies $Kan(m, j)$ for all $0 \leq j \leq m, m \geq 1$ and $Kan!(m, j)$ $0 \leq j \leq m$, $m > n$

- A Lie groupoid is a Lie 1- groupoid in the category (Mfld,$\mathcal{T}$) where covers are surjective submersions.

Kan fibrations

Let $f : X \to Y$ be a morphism of simplicial objects in $(\mathcal{C}, \mathcal{T})$. We say f satisfies $\mathbf{Kan}(n, j)$[or $\mathbf{Kan!}(n, j)$] if the presheaf $\hom(\Lambda^n_j \to \Delta^n, X \to Y)$ is representable and the natural map below is a cover[or an isomorphism].

$$X_n = \hom(\Delta^n, X) \to \hom(\Lambda^n_j \to \Delta^n, X \to Y)$$

- f is a **Kan fibration** if it satisfies $\mathbf{Kan}(n, j)$, $\forall m \geq 1,\ 0 \leq j \leq m$.
- X satisfies $\mathbf{Kan}(n, j)$, iff $f : X \to \mathbb{1}$ satisifes $\mathbf{Kan}(n, j)$.

n-groupoid Kan fibrations

Definition

Let X, Y be n-groupoid objects in $(\mathcal{C}, \mathcal{T})$. A morphism between the n-groupoids $f : X \to Y$ is called an **n-groupoid Kan fibration** if f is a Kan fibration and furthermore the unique Kan conditions $Kan!(n, k)$ for $0 \le k \le n$ are satisfied.

Groupoid actions

Actions of a groupoid

Unlike group actions, in the case of groupoids, we need to specify which groupoid elements can act on a particular element in the set.

Definition

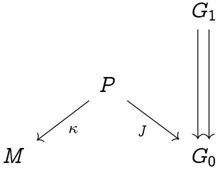
Let $G_1 \rightrightarrows G_0$ be a groupoid and X be a object in $\mathcal{C}$. A right G-action, on X is given by two maps, the anchor map $J : X \to G_0$ and a map $\mu : X \times_{G_0} G_1 \to X$, defined on pairs (x, g) such that $J(x) = t(g)$, written as $\mu(x, g) = x.g$ subject to the following conditions:

- $J(x.g) = s(g), \qquad x \in G_0, g \in G_1$
- $x.\mathbb{1}_{J(x)} = x, \qquad x \in G_0$
- $(x.g).h = x.(gh), \qquad g, h \in G_1, s(g) = t(h)$

Principal groupoid bundles

Definition

A right G-bundle over a manifold M is a manifold P with a right G action and invariant map $\kappa : P \to M$.

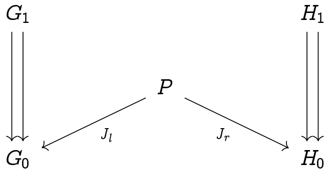


P is principal if

- κ is a surjective submersion.
- the map $(pr_1, \mu) : P \times_{G_0,t} G_1 \to P \times_M P$, sending $(p, g) \mapsto (p, pg)$ is a diffeomorphism.

Lie groupoid bibundles

Let G and H be two Lie groupoids. A $G-H$ bibundle is a manifold having two bundle structures such that the corresponding actions commute. There is a right H-bundle structure and also a left G-bundle structure that commutes.



Hilsum-Skandalis Morphisms

Definition

A Hilsum-Skandalis bibundle is a bibundle P between G and H such that the right H action is principal.

- Given any groupoid map, $f : G \to H$, one can create an HS bibundle.
- HS maps give a good generalization of maps between Lie groupoids.
- If in addition, P is also a left principal G bundle, we say G and H are Morita equivalent.

Why do we care?

- The category of principal G-bundles of a Lie groupoid is a differentiable stack.
- It has been shown that there exists a $1-1$ correspondence between 'Morita equivalent' Lie groupoids and differentiable stacks[BX06].
- Later on, it was proved that the 2 generalizations of Lie groupoids, namely stacky Lie groupoids, and Lie-2- groupoids are in a $1-1$ correspondence [Zhu09].

Action groupoid

If the Lie groupoid $G_1 \rightrightarrows G_0$ acts on a manifold E, one can create the action groupoid A such that

$$A_0 := E \qquad ; \qquad A_1 := E \times_{G_0,t} G_1$$

and there exists a natural projection $\pi : A \to G$.

Remark

The map $\pi : A \to G$ is a $1-$ groupoid Kan fibration.

n-groupoid actions

Definition

The action of an n-groupoid object G in $(\mathcal{C}, \mathcal{T})$ is an n-groupoid Kan fibration $\pi : A \to G$.

Definition

Let $\pi : A \to G$ be an n-groupoid Kan fibration, giving an action of G on $Fib(\pi)$. Let N be an object in $\mathcal{C}$ with a map $\kappa : A \to sk_0(N)$. We call $Fib(\pi)$, a G-bundle over N. If in addition, if κ is acyclic, we call $Fib(\pi)$ a principal G-bundle over N.

Future directions

- [Li15] has worked out the details of actions, principal bundles and bibundles in the case of 2-groupoids.
- A finite data description of Kan fibrations was also given.
- One of our objectives is to work out the details of a bibundle for a general n-groupoid and finitely describe n-groupoid Kan fibrations.

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Thank You!