Coloring Higher Bibundles

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Introduction

Groupoids

Definition

A groupoid is a category where all the morphisms are invertible.

Examples

- A group G is a groupoid with a single object.
- Any equivalence relation is a groupoid.
- Fundamental groupoid of a topological space.
- Lie groupoids in the category Mfld.

Actions of a groupoid

Definition

Let $G_1 \rightrightarrows G_0$ be a groupoid and $X \in obj(\mathcal{C})$. A right G-action, on X is given by,

- the *anchor map* $J : X \to G_0$
- the action map $\mu : X \times_{G_0} G_1 \to X$ with $\mu(x, g) = x.g$

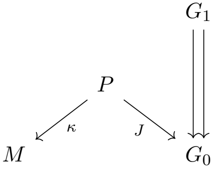
subject to the following conditions:

- $J(x.g) = s(g), \qquad x \in G_0, g \in G_1$
- $x.\mathbb{1}_{J(x)} = x, \qquad x \in G_0$
- $(x.g).h = x.(gh), \qquad g, h \in G_1, s(g) = t(h)$

Principal groupoid bundles

Definition

A right G-bundle over a manifold M is a manifold P with a right G action and invariant map $\kappa : P \to M$.

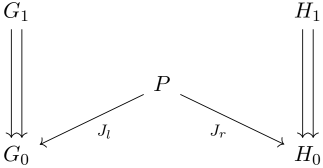


P is principal if

- κ is a surjective submersion.
- the map $(pr_1, \mu) : P \times_{G_0,t} G_1 \to P \times_M P$, sending $(p, g) \mapsto (p, pg)$ is a diffeomorphism.

Lie groupoid bibundles

Let G and H be two Lie groupoids. A $G - H$ bibundle is a manifold having two bundle structures such that the corresponding actions commute. There is a right H-bundle structure and also a left G-bundle structure that commutes.



Hilsum-Skandalis Morphisms

Definition

A Hilsum-Skandalis bibundle is a bibundle P between G and H such that the right H action is principal.

- Given any groupoid map, $f : G \to H$, one can create an HS bibundle.
- If in addition, P is also a left principal G bundle, we say G and H are Morita equivalent.

Examples

- Let G be a Lie groupoid and M a manifold. A (HS) bibundle between $\underline{M}$, the trivial groupoid and G is a (principal) G- bundle over M.
- The G-action on a manifold X gives a bibundle between $*$, the one-object trivial groupoid, and G.

What do we know so far?

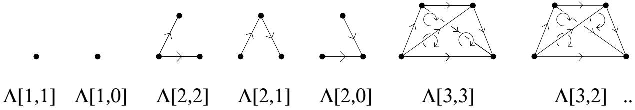
- The category of principal G-bundles of a Lie groupoid is a differentiable stack.
- It has been shown that there exists a $1-1$ correspondence between 'Morita equivalent' Lie groupoids and differentiable stacks[BX06].
- Later on, it was proved that the two generalizations of Lie groupoids, namely stacky Lie groupoids, and Lie-2- groupoids are in a $1-1$ correspondence [Zhu09].

Higher groupoids

Simplicial picture

Notations

Let Δ^n denote the simplicial n-simplex and Λ^n_k or $(\Lambda[n,k])$ denotes the horn



Remark

If you consider the points as objects and the arrows as morphisms in a category, we can see that $\Lambda[2,1]$ gives a pair of composable maps.

Kan conditions

We say the simplicial set X satisfy $Kan(m, j)$ if any map $\Lambda^m_j \to X$ extends to a map $\Delta^m \to X$. In other words $\mathrm{Hom}(\Delta^m, X) \to \mathrm{Hom}(\Lambda^m_j, X)$ is surjective.

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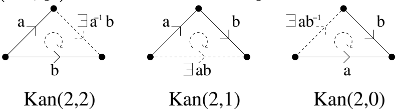
Definition

A **Kan simplicial set** is a simplicial set X which satisfies $Kan(m,j)$ for all m and $j \le m$.



If the extension associated to $Kan(m,j)$ is unique, then we denote it as $Kan!(m,j)$.

n-groupoids

Definition

An *n-groupoid* X is a simplicial set which satisfies $Kan(m, j)$ for all $0 \leq j \leq m$, $m \geq 1$ and also satisfies $Kan!(m, j)$ for all $0 \leq j \leq m$, $m > n$

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- The nerve of a groupoid is a 1- groupoid.
- Crossed modules of group forms a 2-group.

Kan fibrations

Let $f : X \to Y$ be a map of simplicial sets. We say f satisfies **Kan**(n, j)[or **Kan!**(n, j)] the natural map below is a surjection[or an isomorphism].

$$X_n = \hom(\Delta^n, X) \to \hom(\Lambda^n_j \to \Delta^n, X \to Y)$$

Kan fibrations

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- f is a **Kan fibration** if it satisfies **Kan**(n, j), $\forall n \geq 1,\ 0 \leq j \leq n$.
- X satisfies **Kan**(n, j), iff $f : X \to \mathbb{1}$ satisfies **Kan**(n, j).

Replacing the horn Λ^m_j with the boundary $\partial\Delta^m$, we obtain the Acyclic conditions $Acyc(m)$.

n-groupoid Kan fibrations

Definition

Let X, Y be n-groupoids . A morphism $f : X \to Y$ is called an **n-groupoid Kan fibration** if f is a Kan fibration and satisfies $Kan!(n, k)$ for $0 \leq k \leq n$.

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Definition

Let $f : X \to Y$ be a higher groupoid Kan fibration. The *fibre* of f is the simplicial set $\mathrm{Fib}(f)$,

$$\mathrm{Fib}(f)_k := Hom(\Delta^k \to \Delta^0, X \to Y)$$

Action groupoid

If the Lie groupoid $G_1 \rightrightarrows G_0$ acts on a manifold E, one can create the action groupoid A such that

$$A_0 := E \qquad ; \qquad A_1 := E \times_{G_0,t} G_1$$

and there exists a natural projection $\pi : A \to G$.

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Remark

The map $\pi : A \to G$ is a $1-$ groupoid Kan fibration.

n-groupoid actions

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Definition

Let $\pi : A \to G$ be an n-groupoid Kan fibration, giving an action of G on $Fib(\pi)$. Let N be an object in $\mathcal{C}$ with a map $\kappa : A \to sk_0(N)$. We call $Fib(\pi)$, a G-bundle over N. If in addition, if κ is acyclic, we call $Fib(\pi)$ a principal G-bundle over N.

Bibundles

Bimodules between categories

Let $\mathcal{A}, \mathcal{B}$ be categories, a bimodule between $\mathcal{A}$ and $\mathcal{B}$, P is given by a profunctor $\Psi : \mathcal{B}^{op} \times \mathcal{A} \to Sets$.

$$P = \coprod_{A \in \mathcal{A}_0, B \in \mathcal{B}_0} \Psi(B, A)$$

P naturally has an endowed left $\mathcal{A}$ action and right $\mathcal{B}$ action.

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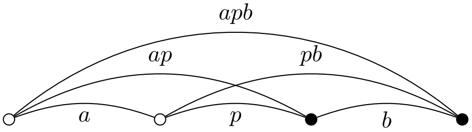
P naturally has an endowed left $\mathcal{A}$ action and right $\mathcal{B}$ action. P can be considered as the diagram below.



Cograph of a Bimodule

Given a bimodule P between categories $\mathcal{A}$ and $\mathcal{B}$, we define the cograph Γ as the category

$$\Gamma_0 = \mathcal{A}_0 \sqcup \mathcal{B}_0 \qquad\qquad \Gamma_1 = \mathcal{A}_1 \sqcup P \sqcup \mathcal{B}_1$$

Example (Join of categories)

Given $\mathcal{A}, \mathcal{B}$, set $P = \mathcal{A}_0 \times \mathcal{B}_0$. Then the cograph is the category obtained from $\mathcal{A}_0 \sqcup \mathcal{B}_0$, with unique arrow $B \to A$ for every $(A, B) \in \mathcal{A}_0 \times \mathcal{B}_0$, called the join of categories, $\mathcal{A} \star \mathcal{B}$.

Note: Nerve preserves joins. $\mathcal{N}(\mathcal{A} \star \mathcal{B}) = \mathcal{N}(\mathcal{A}) \star \mathcal{N}(\mathcal{B})$

Bimodule and Correspondence

Given a cograph Γ of a bimodule , we have a functor $\Gamma \to \mathbb{1}$, where $\mathbb{1}$ is the interval category, sending $\mathcal{A}, \mathcal{B}$ to $0, 1$ and P to the unique arrow $1 \to 0$. Furthermore

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Theorem

The category of $\mathcal{A} - \mathcal{B}$ bimodules is isomorphic to the subcategory of $\mathsf{Cats}/\mathbb{1}$ where the two ends are $\mathcal{A}$ and $\mathcal{B}$.

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The nerve of the cograph decomposes as follows

$$N_n\Gamma = \bigsqcup_{i+j+1=n} N_{i,j}\Gamma, \quad i, j \geq -1,$$

where,

$$N_{i,j}\Gamma = \underbrace{\mathcal{A}_1 \times_{\mathcal{A}_0} \cdots \times_{\mathcal{A}_0} \mathcal{A}_1}_{i \text{ copies}} \times_{\mathcal{A}_0} P \times_{\mathcal{B}_0} \underbrace{\mathcal{B}_1 \times_{\mathcal{B}_0} \cdots \times_{\mathcal{B}_0} \mathcal{B}_1}_{j \text{ copies}},$$

Lie groupoid bibundles

In the case of Lie groupoids, since the source and target maps are surjective submersions, $\mathcal{N}_{i,j}\Gamma$ is a manifold and hence $\mathcal{N}\Gamma$ is a simplicial manifold. Thus we have the theorem

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The category of bibundles between Lie groupoids G and H is isomorphic to the category of categories in Mfd *over* $\mathbb{1}$ *where the two ends are G and H.*

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Theorem

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Remark

The map $\mathcal{N}\Gamma \to \Delta^1 = \mathcal{N}\mathbb{1}$ is an inner Kan fibration.

Hilsum-Skandalis Bibundles

Theorem

Let G and H be Lie groupoids. The category of G-H HS bibundles is isomorphic to the category of categories Γ in Mfd *over $\mathbb{1}$ with the two ends given by G and H such that $N\Gamma \to \Delta^1$ satisfies $Kan(1,0)$ and $Kan!(n,0)$ for $n \geq 2$.*

Remark

The G, H and biaction groupoid are respectively given by the nerves $\mathcal{N}_{i,0}\Gamma$s, $\mathcal{N}_{0,i}\Gamma$s, and $\mathcal{N}_{i,i}\Gamma$ respectively for $i \geq 0$.

Coloring Higher Bibundles

Augmented Bisimplicial sets

The category of augmented bisimplicial sets is given by the functor category $[\Delta_+^{op} \times \Delta_+^{op}, \mathsf{Sets}]$.
There exists a functor

$$T : \mathsf{biSSet}_+ \to \mathsf{SSet}_+$$
$$X \longmapsto \{[n] \mapsto \coprod_{i+j+1=n} X_{i,j}\}$$

called the *total augmented simplicial set* functor.

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Theorem

The functor $T : \mathsf{biSSet}_+ \to \mathsf{SSet}_+/\Delta_*^1$ is an isomorphism of categories.

Colored simplicial complex

We can consider a bisimplicial set (simplicial set over the interval)as a coloured simplicial set with vertices colored by 0 or 1.

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Definition

For fixed $i, j \geq 0$, set $m = i + j - 1$ and let $\Delta^m \xrightarrow{\chi_{i,j}} \Delta^1$ be the canonical map. Then we define the colored horns and simplices.

$$\mathrm{Hom}(\Lambda^m_k[i,j], X) := \mathrm{Hom}_{/\Delta^1}(\Lambda^m_k, X)$$
$$\mathrm{Hom}(\Delta^m[i,j], X) := \mathrm{Hom}_{/\Delta^1}(\Delta^m, X)$$

Colored Kan conditions

Definition

A colored simplicial object $\pi\colon X \to \Delta^1$ satisfies the *colored Kan condition* $\mathrm{Kan}(m,k)[i,j]$ if $\mathrm{Hom}(\Delta^m[i,j], X)$ and $\mathrm{Hom}(\Lambda^m_k[i,j], X)$ are presentable and the map

$$\mathrm{Hom}(\Delta^m[i,j], X) \to \mathrm{Hom}(\Lambda^m_k[i,j], X)$$

is a surjection. If this map is an isomorphism, then we say that $X \to \Delta^1$ satisfies the *colored unique Kan condition* $\mathrm{Kan!}(m,k)[i,j]$.

Bibundle of higher groupoids

Definition

Let X and Y be n-groupoids. A *bibundle* between X and Y is an augmented bisimplicial object Γ with $C_{-1}\Gamma = X_*$ and $R_{-1}\Gamma = Y_*$, such that $T\Gamma \to \Delta^1$ satisfies

- Kan(m, k) for $m \geq 2$ and $0 < k < m$
- Kan$(m, 0)[i, j]$ for $i \geq 2$
- Kan$(m, m)[i, j]$ for $j \geq 2$

moreover, when $m > n$ the unique version of all prescribed Kan conditions hold.

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- $\mathrm{Kan}(m,k)$ for $m \geq 2$ and $0 < k < m$
- $\mathrm{Kan}(m,0)[i,j]$ for $i \geq 2$
- $\mathrm{Kan}(m,m)[i,j]$ for $j \geq 2$

moreover, when $m > n$ the unique version of all prescribed Kan conditions hold.

Remark

If, in addition, $\mathrm{Kan}(m,0)$holds for $m \geq 1$, and when $m > n$ the unique version of all desired Kan conditions hold, then we call such a bibundle *right principal*.

Why do(should) we care about these ?

- [Li15] has worked out the details of actions, principal bundles and bibundles in the case of 2-groupoids and shown correspondence with categorified groupoids.
- For a strict 2-Lie group Γ, Principal Γ-2-bundles are equivalent to Γ-*bundle gerbes*. Moreover they form a 2-stack over the smooth base manifold.
- If Γ is taken to be the Automorphism 2-group of a Lie group, the Principal Γ-2-bundles give the notion of *non-abelian gerbes*[NW11].
- The 3-connected cover of the $Spin(n)$ group, $String(n)$, can be given a smooth 2-group structure[BSCS07].

General Setup

Singleton Grothendieck Pretopolgy

A collection of morphisms $\mathcal{T}$ is called a singleton Grothendieck pretopolgy in $\mathcal{C}$ if

- every isomorphisms in $\mathcal{C}$ is in $\mathcal{T}$.
- the pullback of every morphism in $\mathcal{T}$ exists and is in $\mathcal{T}$.
- if $f, g \in \mathcal{T}$ are composable, then $f \circ g \in \mathcal{T}$.

Morphisms in $\mathcal{T}$ are called covers in $\mathcal{C}$.

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Examples

- Category of smooth manifolds with surjective submersions.
- Category of topological spaces with surjective etale maps.

modified Kan conditions

In the setup of category with a singleton grothendieck pretopology, we modify our Kan conditions as given:

$Kan(m, j)$: The map $\hom(\Delta m, X) \to \hom(\Lambda^m_j, X)$ is a cover in $(\mathcal{C}, \mathcal{T})$.
$Kan!(m, j)$: The map $\hom(\Delta m, X) \to \hom(\Lambda^m_j, X)$ is an isomorphism in $\mathcal{C}$.

n-groupoids in $(\mathcal{C}, \mathcal{T})$

Definition

An n-groupoid object X $(n \in \mathbb{N} \cup \infty)$ is a simplicial object in $(\mathcal{C}, \mathcal{T})$ which satisfies $Kan(m, j)$ for all $0 \leq j \leq m, m \geq 1$ and $Kan!(m, j)$ $0 \leq j \leq m,\ m > n$.

- A Lie groupoid is a Lie 1- groupoid in the category (Mfld,$\mathcal{T}$) where covers are surjective submersions.

Bisimplicial objects in $(\mathcal{C}, \mathcal{T})$

Analogous to the previous cases, we would require bibundles of $n-$groupoid objects to be simplicial objects over the interval category. That is, we need something similar to

Theorem?

The functor $T : \mathsf{biS}\mathcal{C}_+ \to \mathsf{S}\mathcal{C}_+/\Delta^1_*$ is an isomorphism of categories.

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Sadly, this doesn't go through !!!

Extensive categories

Definition

An *extensive category* is a categry $\mathcal{C}$ with finite coproducts such that for each pair $A, B \in obj(\mathcal{C})$ the coproduct induces an isomorphism of categories

$$\mathcal{C}/A \times \mathcal{C}/B \to \mathcal{C}/(A \sqcup B).$$

Correspondence in (C, T)

Let $\mathcal{C}$ be an extensive category with a terminal object $\mathbf{1}$.
Let $j : \mathsf{FinSet} \to \mathcal{C}$ be the functor sending $S \mapsto S \otimes \mathbf{1}$

Theorem

Let T be the functor defined by

$$\mathsf{biS}\mathcal{C}_+ \to \mathsf{S}\mathcal{C}_+, \qquad X \mapsto \left\{ [n] \mapsto \coprod_{i+j+=n} X_{i,j} \right\}.$$

It induces an isomorphism of categories

$$\mathsf{biS}\mathcal{C}_+ \cong \mathsf{S}\mathcal{C}_+/j(\Delta^1_*).$$

Future directions

- Our aim is to provide a nice enough category where the higher groupoids can reside in.
- Verify that these higher bibundles is a good categorical generalization of the usual groupoid bibundles.

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Thank You!