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YONEDA LEMMA

CMIT talk series

- KALIN KRISHNA

"Tell me who your friends are, and I'll tell you who you are"

Theme

- One can identify and understand an object by studying the relations of that object to others.

"identify" "relations" "object"

Recall

- A category '$\mathcal{C}$' consists of two collections $Obj(\mathcal{C})$ & $Mor(\mathcal{C})$

$$X \circlearrowleft 1_X \quad , \quad \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \arrow[r, "h"] & C \end{tikzcd}$$

$A, B, C, X \in Obj(\mathcal{C})$

$f, g \in Hom(A,B)$; $h \in Hom(B,C)$

- A functor $F: \mathcal{C} \to \mathcal{D}$ sends objects to objects & morphisms to morphisms

$$\left(A \xrightarrow{f} B\right) \xmapsto[\text{covariant}]{F} \left(F(A) \xrightarrow{F(f)} F(B)\right)$$

$$\left(A \xrightarrow{f} B\right) \xmapsto[\text{contravariant}]{F} \left(F(B) \xrightarrow{F(f)} F(A)\right)$$

satisfying $F(1_X) = 1_{F(X)}$, $F(g \circ f) = F(g) \circ F(f)$ (or $F(g \circ f) = F(f) \circ F(g)$)

- A natural transformation $\eta: F \Rightarrow G$, between functors $F, G: \mathcal{C} \to \mathcal{D}$ is an assignment $Obj(\mathcal{C}) \longrightarrow Mor(\mathcal{D})$

ie

$\forall$ $A \in Obj(\mathcal{C})$ $\eta_A : F(A) \longrightarrow G(A)$ $\in \mathcal{D}$.

Given $A \xrightarrow{f} B \in \mathcal{C}$. η satisfies

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\circlearrowright"] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\eta_B"'] & G(B) \end{tikzcd}$$

- One can identify and understand an object by studying the relations of that object to others.

For any object X, we consider all the maps into X

'$Hom(-, X)$'

This assignment is functorial. That is

$$h_X := Hom(-, X) : \mathcal{C}^{op} \longrightarrow Set$$
$$A \longmapsto Hom(A, X)$$

What about morphisms?

$$\text{Let } (A \xrightarrow{f} B) \overset{h_X}{\longmapsto} \left(Hom(B, X) \xrightarrow{h_X^*(f)} Hom(A, X) \right)$$

given $\alpha \in Hom(B, X)$ $\quad h_X^*(f)(\alpha) := \alpha \circ f \in Hom(A, X)$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, dashed, "\alpha \circ f"'] & B \arrow[d, "\alpha"] \\ & X \end{tikzcd}$$

So given any $X \in Obj(\mathcal{C})$, we have a contravariant functor to Set

Denote $\hat{\mathcal{C}} := Func[\mathcal{C}^{op}, Set]$

Then we get a functor

$$h : \mathcal{C} \longrightarrow \hat{\mathcal{C}}$$
$$X \longmapsto h_X$$

What about in the level of morphisms?

Let $X \xrightarrow{f} Y$. We need a natural transformation $\eta^f : h_X \longrightarrow h_Y$

ie Given an object $A \in \mathcal{C}$, we need a map $\eta_A : h_X(A) \longrightarrow h_Y(A)$

Define η^f_A : $\mathrm{Hom}(A,X) \longrightarrow \mathrm{Hom}(A,Y)$

$$\alpha \longmapsto f\circ\alpha$$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ A \arrow[u, "\alpha"] \arrow[ur, dashed] & \end{tikzcd}$$

So $\eta^f_A = f\circ\_$

Is it natural transformation?

Let $A \xrightarrow{g} B$, we need

$$\begin{tikzcd} h_X(A) \arrow[r, "\eta^f_A"] & h_Y(A) \\ h_X(B) \arrow[u, "h_X(g)"] \arrow[r, "\eta^f_B"'] & h_Y(B) \arrow[u, "h_Y(g)"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Hom}(A,X) \arrow[r, "f\circ\_"] & \mathrm{Hom}(A,Y) \\ \mathrm{Hom}(B,X) \arrow[u, "\_\circ g"] \arrow[r, "f\circ\_"'] & \mathrm{Hom}(B,Y) \arrow[u, "\_\circ g"'] \end{tikzcd}$$

β

$$\boxed{f\circ\beta\circ g = f\circ\beta\circ g}$$

So η^f is indeed a natural transformation.

In short we have a functor $h : \mathcal{C} \longrightarrow \hat{\mathcal{C}}$ which sends each object to the corresponding Hom functor

i.e. we have got a way to relate an object to its relations

Qn: Can we identify an object X with h_X ???

what do we need, the relations X hold with other objects should reflect on $\hat{\mathcal{C}}$ with h_X.

ie if we are given a map $X \to Y$, there should exist one and only one morphism from $h_X \Rightarrow h_Y$

ie $\quad \mathrm{Hom}(X,Y) \longrightarrow \mathrm{Nat}(h_X, h_Y)$ is bijection.

ie the Functor h is Fully faithful

Given two distinct $f, g \in \mathrm{Hom}(X,Y)$, we can see that $\eta^f, \eta^g \in \mathrm{Nat}(h_X, h_Y)$ are distinct Check.

Conversely, if we are given a natural transformation $\eta \in \mathrm{Nat}(h_X, h_Y)$ does it arise from a map $X \to Y$?

YES! says Yoneda Lemma.

So what does it say?

$$\mathrm{Nat}(h_X, h_Y) \cong \mathrm{Hom}(X,Y)$$

Observing that $h_Y : \mathcal{C}^{op} \to \mathrm{Set}$ is a Functor, Above statement can be written as

$$\mathrm{Nat}(h_X, h_Y) \cong h_Y(X)$$

What is more surprising is that, Yoneda lemma says something more.

It says that for any contravariant functor $F : \mathcal{C}^{op} \to \mathrm{Set}$ the result is true!

YONEDA LEMMA

For any functor $F : \mathcal{C}^{op} \to \mathrm{Set}$, and object $X \in \mathrm{Obj}(\mathcal{C})$, we have a bijection $\mathrm{Nat}(h_X, F) \cong F(X)$ of sets.

We can also have a covariant version considering the functor $k^X : \mathcal{C} \longrightarrow Set$

$$A \longmapsto Hom(X, A)$$

So Lemma becomes:

For any functor $G : \mathcal{C} \to Set$ and any object $X \in Obj(\mathcal{C})$

$$Nat(k^X, G) \cong G(X)$$

Remark:- The isomorphism in Lemma is natural with respect to both X and F.

Corollary: The functor $h : \mathcal{C} \longrightarrow \mathcal{C}^\wedge$ is fully faithful.

ie, given X, Y

$$Hom(X, Y) \cong Nat(h_X, h_Y)$$

Corollary: $X \simeq Y$ if and only if $h_X \cong h_Y$

EXAMPLES

- Yoneda Lemma in Category of matrices

Every naturally defined row (column) operation is obtained by left (right) multiplication by the matrix obtained by applying the operation to identity matrix.

Let Mat denote the category of matrices, defined as

$$Obj(Mat) = \{1, 2, 3, \dots\} = \mathbb{N}$$

$$Hom(n, m) = \{\, m \times n \text{ matrices} \} \qquad n \xrightarrow{A} m \Rightarrow A_{m \times n}$$

A k-row functor is a functor $h_k : Mat^{op} \longrightarrow Set$

$$h_k(n) = \{\, k \times n \text{ matrices} \} = Hom(n, k)$$

$$\left(n \xrightarrow{A} m\right) \longmapsto^{h_k} \{k\times m \text{ matrices}\} \xrightarrow{A} \{k\times n \text{ matrices}\}$$

$$X_{k\times m} \longmapsto (XA)_{k\times n}$$

A naturally defined row operation on row functors is a natural transformation $\eta : h_k \longrightarrow h_j$

i.e, given $n \in Obj(Mat)$ $\eta_n : h_k(n) \longrightarrow h_j(n)$

that is η_n sends $k\times n$ matrices to $j\times n$ matrices

with naturality, So if $n \xrightarrow{A} m$

$$\begin{tikzcd} h_k(n) \arrow[r, "\eta_n"] & h_j(n) \\ h_k(m) \arrow[u, "h_k(A)"] \arrow[r, "\eta_m"'] \arrow[ur, phantom, "\circlearrowright"] & h_j(m) \arrow[u, "h_j(A)"'] \end{tikzcd}$$

Note that left multiplication by a $j\times k$ matrix will give a natural transformation $h_k \longrightarrow h_j$

By Yoneda Lemma

$$Nat(h_k, h_j) \cong Hom(k, j)$$

Thus all the natural transformations $h_k \Longrightarrow h_j$ arise as left multiplication by a $j\times k$ matrix.

• Proof of CAYLEY's THEOREM

CAYLEY's THM

Any Group G is isomorphic to a subgroup of symmetric group on G.

Let G be group. Consider $\overline{G}$ as a category

$Obj(\overline{G}) = \{\bullet\}$ $\quad\quad$ $Mor(\overline{G}) = G$

A functor $F : \overline{G}^{op} \to Set$ is a set X with right G action where $X = F(\bullet)$

consider the Yoneda functor $h_\bullet = Hom(-, \bullet)$

Note that $h_\bullet$ gives G itself ($h_\bullet(\bullet) = Hom(\bullet, \bullet) = Mor\,\overline{G} = G$) as a G-set

Yoneda lemma says $Nat(h_\bullet, h_\bullet) \cong h_\bullet(\bullet) = G$

What are natural transformations $\eta : h_\bullet \Rightarrow h_\bullet$

$\eta : G \to G$, given any $g \in G$

$$\begin{tikzcd} G \arrow[r, "\eta"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & G \arrow[d, "g"] \\ G \arrow[r, "\eta"'] & G \end{tikzcd} \quad \Longrightarrow \quad \begin{array}{l} \text{for any } h \in G \\ \eta(h) \cdot g = \eta(h \cdot g) \\ \text{ie, } \eta \text{ is } G\text{-equivariant} \end{array}$$

So natural transformations are G-equivariant maps $G \to G$

But from Yoneda lemma, this corresponds to right multiplication by a fixed group element. Thus an automorphism.

Thus we get an isomorphism between G and automorphism group of right G-set G and hence a subgroup of symmetry group.

- Tensoring is commutative in category of Vector spaces.

Note that $Bilin(V, W; U) \cong Bilin(W, V; U)$ $\forall U \in Vect$

But $\mathrm{Bilin}(V,W;U) \cong \mathrm{Hom}(V\otimes W, U)$

$\Rightarrow \mathrm{Hom}(V\otimes W, U) \cong \mathrm{Bilin}(V,W;U) \cong \mathrm{Bilin}(W,V;U) \cong \mathrm{Hom}(W\otimes V, U)$

$\Rightarrow k^{V\otimes W} \cong k^{W\otimes V} \Rightarrow V\otimes W \cong W\otimes V$

Yoneda Lemma: Proof

What do we have to prove?

Given $X \in \mathrm{Obj}(\mathcal{C})$ and a functor $F : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$

$$\boxed{\mathrm{Nat}(h_X, F) \cong F(X)} \quad \text{Naturally in } X, F$$

- We construct a map $\Delta_{X,F} : \mathrm{Nat}(h_X, F) \longrightarrow F(X)$

  Let $\alpha \in \mathrm{Nat}(h_X, F)$ be a natural transformation.

  ie given any $A \in \mathrm{Obj}(\mathcal{C})$, we have $\alpha_A : h_X(A) \to F(A)$

  in particular consider $\alpha_X : h_X(X) = \mathrm{Hom}(X,X) \longrightarrow F(X)$

  Define $\Delta_{X,F}(\alpha) = \alpha_X(\mathbb{1}_X)$

- We construct a map $\tilde{\Delta}_{X,F} : F(X) \longrightarrow \mathrm{Nat}(h_X, F)$

  Let $x \in F(X)$, $\tilde{\Delta}_{X,F}(x)$ should be a nat trans $: h_X \Rightarrow F$

  i.e, given $A \in \mathrm{Obj}(\mathcal{C})$ $\tilde{\Delta}_{X,F}(x)_A : h_X(A) \longrightarrow F(A)$

  Let $f \in h_X(A)$, that is $A \xrightarrow{f} X \Rightarrow F(X) \xrightarrow{F(f)} F(A)$

  Define $\tilde{\Delta}_{X,F}(x)_A(f) := F(f)(x)$

Exercises (Not for the faint-hearted !!!)

① Show that
$$\Phi : \mathcal{C}^{op} \times \mathcal{C}^{\wedge} \longrightarrow \mathrm{Set}$$
$$(X, F) \longmapsto F(X)$$
is a Functor.

② Show that
$$\Psi : \mathcal{C}^{op} \times \mathcal{C}^{\wedge} \longrightarrow \mathrm{Set}$$
$$(X, F) \longmapsto \mathrm{Nat}(h_X, F)$$
is a Functor.

③ Show that $\tilde{\Delta} : \Phi \Longrightarrow \Psi$ is a natural transformation

④ Show that $\Delta : \Psi \Rrightarrow \Phi$ is a natural transformation.

⑤ Show that $\Delta \circ \tilde{\Delta} = \mathbb{1}$, $\tilde{\Delta} \circ \Delta = \mathbb{1}$.

If you solved all, then Voila, Q.E.D

REFERENCES

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Thank You

Stay Safe!!!